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PROCEEDINGS

OF THE

PHYSICAL SOCIETY OF LONDON.

From December 1912 *to August* 1913.

VOL. XXV.

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1912-1913.

October 25th, 1912.

Meeting held at the Imperial College of Science.

Prof. C. H. **Lees**, F.R.S., Vice-President, in the Chair.

The following Papers were read :—

1. On the Constitution of the Mercury Lines Examined by an Echelon Grating and a Lummer Gehrcke Plate. By Prof. H. **Nagaoka** and Mr. T. **Takamine**.
2. Note on the Mutual Inductance of Two Coaxial Circular Currents. By Prof. H. **Nagaoka**.
3. The Absorption of Gases in Vacuum Tubes. By Mr. S. E. **Hill**.

November 8th, 1912.

Meeting held at the Imperial College of Science.

Prof. A. **Schuster**, F.R.S., President, in the Chair.

The following Papers were read :—

1. On a Method of Measuring the Thomson Effect. By Mr. H. R. **Nettleton**.
2. An Improved Joule Radiometer and its Applications. By Mr. F. W. **Jordan**.

3. Note on the Attainment of a Steady State when Heat Diffuses along a Moving Cylinder. By Miss A. SOMERS.

4. On the Thermomagnetic Study of Steel. By Dr. S. W. J. SMITH.

November 22nd, 1912.

Meeting held at the Imperial College of Science.

Prof. C. H. LEES, F.R.S., Vice-President, in the Chair.

The following Papers were read :—

1. On a Column Testing Machine. By Prof. E. G. COKER.

2. The Law of Plastic Flow of a Ductile Material and the Phenomena of Elastic and Plastic Strains. By Mr. C. E. LARARD.

Some Kinematograph Illustrations of the Twisting and Breaking of Large Wrought-Iron and Steel Specimens were also exhibited by Mr. C. E. LARARD.

December 17th, 1912.

The Eighth Annual Exhibition of Physical Apparatus was held at the Imperial College of Science from 3 p.m. to 6 p.m. in the afternoon and from 7 p.m. to 10 p.m. in the evening.

At both the afternoon and evening meetings a Discourse was given by Mr. S. G. BROWN on "Some Methods of Magnifying Feeble Signalling Currents."

The following firms exhibited apparatus :—

R. & J. Beck (Ltd.), Cambridge Scientific Instrument Co., C. F. Casella & Co. (Ltd.), Harry W. Cox & Co. (Ltd.), Crompton & Co. (Ltd.), J. H. Dallmeyer (Ltd.), Dubilier Electrical Syndicate (Ltd.), Edison Storage Battery Co., Elliott Brothers, Evershed & Vignoles (Ltd.), Foster Instrument Co., A. Gallenkamp & Co. (Ltd.), Gambrell Brothers (Ltd.), F. Harrison Glew, John J. Griffin & Sons (Ltd.), Kelvin & James White (Ltd.), E. Leitz, Marconi Wireless Telegraph Co. (Ltd.), Nalder Bros. & Thompson (Ltd.), R. W. Paul, Townson & Mercer (Ltd.), Westminster Engineering Co. (Ltd.), Weston Electrical Instrument Co., Alexander Wright & Co. (Ltd.), Carl Zeiss (Ltd.).

January 24th, 1913.

Meeting held at the Imperial College of Science.

Prof. C. H. Lees, F.R.S., Vice-President, in the Chair.

The following Papers were read :—

1. On the Resistance of Electrolytes. By Dr. S. W. J. Smith and Mr. H. Moss.

2. The Electrical Conductivity and Fluidity of Strong Solutions. By Mr. W. S. Tucker.

Annual General Meeting.

February 14th, 1913.

Meeting held at the Imperial College of Science.

Prof. A. Schuster, F.R.S., President, in the Chair

The Report of the Council was read by the Secretary.

Since the last Annual General Meeting there have been held 13 ordinary meetings and one informal meeting. All of these were held at the Imperial College of Science.

An unusual feature of the past Session was a joint meeting with the Optical Convention on the morning and evening of June 21st. Eight Papers dealing with optical subjects were read at this meeting, and a Part of the "Proceedings" devoted exclusively to these Papers has been issued.

The Eighth Annual Exhibition of Apparatus by Manufacturers was held on December 17th in the afternoon and evening, the number of Fellows and visitors present being about 550. An experimental discourse on "Some Methods of Magnifying Feeble Signalling Currents" was kindly given by Mr. S. G. Brown. The number of exhibitors was 27.

The number of Ordinary Fellows now on the roll, as distinct from Honorary Fellows, is 440; 17 new Fellows have been elected, there have been 10 resignations, and 5 Fellows have been struck off the roll for non-payment of subscriptions. The Society has to mourn the loss by death of 7 Fellows—namely, the Earl of Crawford, F.R.S., H. Donaldson, Major-Gen. Festing, A. B. Harding, A. E. Hodgson, J. T. Hurst and Prof. Osborne Reynolds, F.R.S.

The Report was adopted by the meeting.

The Report of the Treasurer and the Balance-Sheet were presented by the Treasurer.

The total income of the Society shows a slight increase over that of 1911. This is mainly due to an increase in the subscriptions and in the sales of publications.

The expenditure for the year has also increased, due to the issue of the special number recording the Papers read at the joint meeting of the Physical Society and Optical Convention.

By comparing the balance brought forward on the 1st January, 1912, £97. 17s. 7d., with the balance carried forward (less two cheques) on the 1st January, 1913—namely, £140. 12s. 2d.—it will be seen that the Society has increased its assets during the year by £42. 14s. 7d. This is a very gratifying result in view of the special expenditure that has been incurred, and I think may be taken as indicating that the finances of the Society are in a very sound condition.

The property account of the Society is practically the same as at the date of the last accounts. The values of our securities have again slightly decreased, but when we consider the great depreciation that has taken place in trustee securities during the last few years, the Society has to congratulate itself that the depreciation of the securities is not more considerable.

The Society has again to thank the Manager of Parr's Bank for kindly valuing the securities at the 31st December, 1912, and supplying the figures which appear in the accounts.

The liabilities on account of the Life Composition Fund have decreased during the year owing to the death of three life Fellows and to no new Fellows having compounded for their subscriptions. As a result of the year the balance available in the General Fund of the Society is practically the same as at the date of last year's accounts.

In my last report I mentioned that the back numbers of the publications of the Society were being re-valued. The work of sorting and getting into order the whole of the back numbers of the "Proceedings" of the Society is now completed, and I have made a valuation of the publications of the Society, which is now returned in the accounts at the revised figure of £240.

The Report of the Treasurer was adopted.

THE TREASURER IN ACCOUNT WITH THE PHYSICAL SOCIETY, FROM JANUARY 1ST, 1912, TO DECEMBER 31ST, 1912.

Dr.	£	s.	d.	£	s.	d.
Entrance Fees	19	19	0			
Subscriptions	501	11	0			
Extra Subscriptions	23	2	0			
Arrears	18	18	0			
Subscriptions paid in Advance	8	8	0			
Students' Subscriptions	5	15	6			
Subscriptions for "Science Abstracts" and Advanced Proofs	14	16	6			
				592	10	0
Composition Fees					nil	
Dividends:—						
Furness Debenture Stock	15	1	2			
Midland Railway	37	13	4			
Metropolitan Board of Works	6	12	0			
Lancaster Corporation Stock	11	6	0			
New South Wales	8	7	6			
London, Brighton & South Coast Railway	23	10	10			
Great Eastern Railway	18	16	8			
				121	7	6
Interest on £400 deposit account				9	2	5
Sales of Publications ("The Electrician" Printing & Publishing Co.)				83	8	10
				806	8	9
Outstanding credit from last account (Sales, etc., 1911)				18	7	5
January 1, 1912, balance at Bank				97	17	7
				£922	13	9

Cr.	£	s.	d.	£	s.	d.
"Science Abstracts"				280	16	6
Printing and Distributing Publications of the Society:—						
"The Electrician" Printing & Publishing Co.—						
Proceedings	212	6	11			
Bulletin	29	6	5			
Distribution	44	6	0			
General	27	19	1			
	313	18	5			
Less credit for use of type	1	19	9			
				311	18	8
Special Optical Convention. Part VI. of Vol. XXIV				51	5	8
Periodicals				12	0	0
Reporting				35	0	0
Refreshments and attendance at meetings				33	15	9
Authors' Expenses				0	17	6
Petty cash				16	2	2
Bank commission & stamped cheques				0	6	2
Subscriptions paid in error				2	7	6
Royal Asiatic Society				2	2	0
International Catalogue				12	2	3
Address to Royal Society				5	0	0
				763	14	2
Outstanding Liability from balance (Exhibition, etc., 1911)				18	7	5
Balance at Bank, January 1, 1913	209	14	0			
Less two Cheques	69	1	10			
				140	12	2
				£922	13	9

WILLIAM DUDDELL, *Honorary Treasurer.*

Audited and found correct,

CHARLES V. DRYSDALE.
HENRY M. ELDER.

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31st, 1912.

Assets.	£	s.	d.
Subscriptions due, Treasurer's estimate...	18	18	0
£533 Furness Ry. Co. 3 per cent. Debenture Stock	389	5	0
£1,600 Midland Railway 2½ per cent. Preference Stock	1,016	0	0
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	197	0	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	316	0	0
£254. 2s. 9d. New South Wales 3½ per cent. Inscribed Stock	242	10	0
£500 London, Brighton & South Coast Railway Ordinary Stock	530	0	0
£500 Great Eastern Railway 4 per cent. Debenture Stock	505	0	0
Balance at Bank	209	14	0
Ditto on deposit..	400	0	0
Publications (Treasurer's Estimate)...............	240	0	0
	£4,064	7	0

Liabilities.	£	s.	d.
Two Cheques..	69	1	10
Life Compositions..	2,129	10	0
Balance General Fund..	1,865	15	2
	£4,064	7	0

WILLIAM DUDDELL, *Honorary Treasurer.*

Audited and found correct,

CHARLES V. DRYSDALE,
HENRY M. ELDER,

LIFE COMPOSITION FUND.

	£	s.	d.
178 Fellows paid £10	1,780	0	0
3 Fellows paid £15	45	0	0
4 Fellows paid £21	84	0	0
7 Fellows paid £31. 10s.	220	10	0
	£2,129	10	0

NOTE.—Three Fellows who paid £10 deceased during year 1912.

Audited and found correct,

WILLIAM DUDDELL, *Honorary Treasurer.*

CHARLES V. DRYSDALE.
HENRY M. ELDER.

The Election of Officers and Council then took place, the new Council being constituted as follows :—

President.—Prof. A. SCHUSTER, Ph.D., F.R.S.

Vice-Presidents, who have filled the Office of President.—Prof. G. C. FOSTER, D.Sc., LL.D., F.R.S. ; Prof. W. G. ADAMS, M.A., F.R.S. ; Prof. R. B. CLIFTON, M.A., F.R.S. ; Prof. A. W. REINOLD, C.B., M.A., F.R.S. ; Prof. Sir ARTHUR W. RÜCKER, M.A., D.Sc., F.R.S. ; Sir W. DE W. ABNEY, R.E., K.C.B., D.C.L., F.R.S. ; Prin. Sir OLIVER J. LODGE, D.Sc., LL.D., F.R.S. ; Prof. SILVANUS P. THOMPSON, D.Sc., F.R.S. ; R. T. GLAZEBROOK, C.B., D.Sc., F.R.S. ; Prof. J. H. POYNTING, M.A., Sc.D., F.R.S. ; Prof. J. PERRY, D.Sc., F.R.S. ; C. CHREE, Sc.D., LL.D., F.R.S. ; Prof. H. L. CALLENDAR, M.A., LL.D., F.R.S.

Vice-Presidents.—Prof. C. H. LEES, D.Sc., F.R.S. ; Prof. T. MATHER, F.R.S. ; A. RUSSELL, M.A., D.Sc. ; F. E. SMITH.

Secretaries.—W. R. COOPER, M.A. ; S. W. J. SMITH, M.A., D.Sc.

Foreign Secretary.—Prof. S. P. THOMPSON, D.Sc., F.R.S.

Treasurer.—W. DUDDELL, F.R.S.

Librarian.—S. W. J. SMITH, M.A., D.Sc.

Other Members of Council.—Prof. C. G. BARKLA, D.Sc., F.R.S. ; Prof. P. V. BEVAN, M.A., Sc.D. ; W. H. ECCLES, D.Sc. ; Prof. J. W. NICHOLSON, M.A., D.Sc. ; Major W. A. J. O'MEARA, C.M.G. ; T. C. PORTER, M.A., D.Sc. ; Prof. the Hon. R. J. STRUTT, F.R.S. ; W. E. SUMPNER, D.Sc. ; R. S. WHIPPLE, R. S. WILLOWS, M.A., D.Sc.

A Paper on the Dynamics of Pianoforte Touch was then read by Prof. G. H. BRYAN, Sc.D., F.R.S.

February 28th, 1913.

Meeting held at King's College.

Prof. A. SCHUSTER, F.R.S., President, in the Chair.

The following Papers were read :—

1. On the Interference of Röntgen Radiation. By Prof. C. G. BARKLA and Mr. G. H. MARTYN.
2. On Alternating-current Magnets. By Prof. E. WILSON.

March 14th, 1913.

Meeting held at University College.

Dr. A. RUSSELL, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. Some Oscillograms of Condenser Discharges and a Simple Theory of Coupled Circuits. By Dr. J. A. FLEMING.
2. On the Stretching and Breaking of Sodium and Potassium. By Mr. BEVAN B. BAKER.
3. On the Latent Heat of Evaporation of Steam from Salt Solutions. By Mr. R. G. LUNNON.
4. On Some Flame Spectra. By Dr. E. N. DA C. ANDRADE.

Some Braun Cathode Ray Tubes used as a high-frequency Oscillograph and an Electrostatic Machine for working them were exhibited by Dr. J. A. FLEMING.

Some Spark Photographs at High Pressure were exhibited by Mr. W. B. HAINES.

April 11th, 1913.

Meeting held at the Imperial College of Science.

Prof. C. H. LEES, F.R.S., Vice-President, in the Chair.

The following Papers were read :—

1. On Errors in Magnetic Testing due to Elastic Strain. By Mr. A. CAMPBELL and Mr. H. C. BOOTH.
2. Note on Cathodic Sputtering. By Dr. G. W. C. KAYE.
3. Vibration Galvanometers with Unifilar Torsional Control. By Mr. A. CAMPBELL.

April 25th, 1913.

Meeting held at the Imperial College of Science.

Prof. C. H. LEES, F.R.S., Vice-President, in the Chair.

The following Papers were read :—

1. A Graphical Method of Optical Imagery. By Mr. W. R. BOWER.
2. On the Spectroscopic Resolution of an Arbitrary Function. By Dr. C. V. BURTON.

May 16th, 1913.

Meeting held at the Imperial College of Science.

Prof. C. H. Lees, F.R.S., Vice-President, in the Chair.

The following Papers were read :—

1. Some Experiments to detect β-Rays from Radium A. By Dr. W. Makower and Dr. S. Russ.
2. On Dust Figures. By Dr. J. Robinson.
3. Vibration Galvanometer Design. By Dr. H. F. Haworth.

May 30th, 1913.

Meeting held at the Imperial College of Science.

Prof. A. Schuster, F.R.S., President, in the Chair.

Prof. A. W. Bickerton gave the Society an account of his theory of the Origin of New Stars.

The following Paper was read :—

On Electro-thermal Phenomena at the Contact of Two Conductors with a Theory of a Class of Radio-Telegraph Detectors. By Dr. W. H. Eccles.

The following Papers were taken as read :—

1. The Extraordinary Ray resulting from the Internal Reflection of an Extraordinary Ray at the Surface of an Uniaxal Crystal. By Mr. James Walker.
2. On the Evaluation of Certain Combinations of the Ber, Bei and Allied Functions. By Mr. S. Butterworth.

June 13th, 1913.

Meeting held at the Imperial College of Science.

Prof. C. H. Lees, F.R.S., Vice-President, in the Chair.

The following Papers were read :—

1. Some Experiments on Tinfoil Contact with Dielectrics. By G. E. Bairsto.

2. On a Method of Measuring the Pressure of Light by means of Thin Metal Foil. By Mr. G. D. West.

3. On the Quantum Theory of Energy and the Emission of Electricity from Hot Bodies. By Dr. W. Wilson.

June 27th, 1913.

An informal Meeting held at the National Physical Laboratory, where a number of demonstrations were given, and Fellows had an opportunity of inspecting the laboratories.

ERRATUM.

Page 217, line 9 from the bottom. *For* ε *read* ε^{jPt}.

I. *The Constitution of Mercury Lines examined by an Echelon Grating and a Lummer-Gehrcke Plate.* By *Prof.*

H. NAGAOKA *and* T. TAKAMINE, *Imperial University, Tokyo.*

RECEIVED SEPTEMBER 4, 1912. READ OCTOBER 25, 1912.

§ 1. *Object of the Investigation.*—The constitution of mercury lines has been investigated by different physicists, but the results are generally not in good agreement. It may be due, on the one hand, to the imperfection of the optical instruments, and, on the other, to the nature of the source of light.

With an echelon spectroscope, the presence of ghosts can hardly be avoided, and the order of the spectrum, for satellites at some distance from the principal line, is extremely ambiguous, due to overlapping.

Although the instrument suffers from these two defects, the intensity of light is far superior to that in other interferential apparatus, as it concentrates the light mostly on one or two orders of spectra. An interesting Paper by Stansfield and Walmsley* sheds some light on the nature of the ghosts to be observed in an echelon grating, but the elimination of the ghosts can be easily effected by crossing the spectra according to the method of Gehrcke and von Baeyer,† which is analogous to that of crossed prisms first used by Kundt in investigating anomalous dispersion. The echelon may be crossed by a grating, or by another echelon, or by a Lummer-Gehrcke plate. We followed the last method to clear some mists, which still hang round the nature of the satellites, and that for reasons which will be explained afterwards.

The intensity of the satellites has not yet been accurately measured nor has the regularity in the position of the satellites been well ascertained. According to our investigation, the intensity as well as the position of the satellites are not altogether irregular; this fact will be of special interest to those who wish to unravel the secrets of atomic structure.

§ 2. *Echelon Grating.*—The echelon grating and the Lummer-Gehrcke plate used in the present investigation were made by Hilger. The former was of the following dimensions :—

Thickness of the plate	9·350 mm.
Number of plates	35
Steps	1 mm.
Length	32·73 cm.

---

* Stansfield and Walmsley, "Phil. Mag.," **23**, p. 25, 1912.

† Gehrcke and v. Baeyer, "Ann. d. Phys.," **20**, p. 267, 1906.

The thickness given by Hilger agrees with that obtained by actual measurement on a piece of prism cut out from the same plate used for constructing the echelon,* by means of Abbe's contact micrometer (Dickenmesser) reading to one micron. It was compared with a nickel-steel étalon previously standardised at the Bureau International des Poids et Mesures. The above prism (refracting angle 60° 0′ 0·9″) was used in finding the indices of refraction of echelon plate for mercury lines with a spectrometer (diameter of the graduated circle 30 cm., reading to 1″ by microscopes placed diametrically opposite). They can be almost exactly expressed either by means of Cauchy's or Hartmann's formula. It is found on interpolation that the numbers given by Hilger for refractive indices differ slightly in the fourth decimal.

The separation of the spectra of consecutive orders in the position of minimum deviation is given by the formula

$$d\lambda_{\text{max.}} = \frac{\lambda^2}{t\left\{(\mu-1) - \lambda\frac{\partial\mu}{\partial\lambda}\right\}}.$$

The value of $d\lambda_{\text{max.}}$ (in Å.U.) and the indices of refraction are as follows :—

λ	5,790·5	5,769·5	5,461·0	4,358·6	4,078·1	4,046·8
μ (15°C.)	1·57456	1·57473	1·57725	1·59181	1·59790	1·59873
$d\lambda_{\text{max.}}$	0·5805	0·5760	0·5090	0·3002	0·2547	0·2497

The homogeneity of the glass, as arranged in the echelon grating, was tested by placing the apparatus between two crossed Nicol prisms on a large Paalzow's polariscope, made by Schmidt and Haensch. The dark field was then dimly lighted, but none of the polarisation band was to be detected, showing that the pile of plates, though not entirely free from strain, was almost homogeneous. That the echelon was slightly strained could at once be proved, since a beam of parallel rays on passing through it did not issue as such, but the pile acted somewhat like a lens. This fact is already well known, so that it is only necessary to note it.

---

* Lunelund (" Ann. d. Phys.," **34**, p. 505, 1911) has some doubt about the thickness of our echelon. The number quoted by Lunelund may refer to another echelon, of which there are several in Japan.

§ 3. *Lummer-Gehrcke Plate.*—The Lummer plate was of the same kind of glass as that of the echelon grating The plate was 20 cm. long, 3·5 cm. wide and 1·09116 cm. thick. The last number was determined by Abbe's contact micrometer. The indices of refraction were determined directly by means of Abbe's crystal refractometer, the constant of the instrument being checked by a quartz plate, and also by the prism of the echelon plate, whose indices of refraction were already measured by another instrument of high accuracy. The indices thus found were less by 7 or 8 in the 4th decimal place from those of the echelon plate; they were conformable to Cauchy's as well as to Hartmann's formula.

The value of $d\lambda_{\text{max.}}$ may be calculated in the following manner. The order of the spectrum h is given by

$$h = \frac{2t\sqrt{\mu^2 - \sin^2 i}}{\lambda}, \qquad (1)$$

where t is the thickness and i the angle of exit.

Differentiating it with respect to λ,

$$h^2\lambda = 4t^2\left(\mu\frac{\partial\mu}{\lambda} - \frac{\sin 2i}{2}\frac{\partial i}{\partial\lambda}\right). \qquad (2)$$

Since h is of the order 5×10^4 in the present experiment, we get the equation of finite difference

$$h\lambda^2 \,.\, \Delta h = -2t^2 \sin 2i \,.\, \Delta i. \qquad (3)$$

Equation (2) gives an approximate relation

$$2t^2 \sin 2i \,.\, \Delta i = \left(h^2\lambda - 4t^2\mu\frac{\partial\mu}{\partial\lambda}\right)\Delta\lambda. \qquad (4)$$

Hence, by putting $\Delta h = 1$, we have $\Delta\lambda = d\lambda_{\text{max.}}$, so that

$$d\lambda_{\text{max.}} = -\frac{\lambda^2}{h\lambda - \frac{4t^2\mu}{h}\frac{\partial\mu}{\partial\lambda}}. \qquad (5)$$

The importance of introducing the correction for dispersion was first recognised by v. Baeyer.*

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* v. Baeyer, "Verh. d. Deutsch. Phys. Ges.," 10, p. 733, 1908.

For grazing exit

$$h_0 = \frac{2t\sqrt{\mu^2-1}}{\lambda}, \quad . \; . \; . \; . \; . \; . \; . \quad (1')$$

and the corresponding $d\lambda_{\text{max.}}$ (in Å.U.) is as follows :—

λ	5,790	5,769	5,461	4,359	4,078	4,047
h_0	45,820	46,000	48,710	61,980	66,720	67,200
$d\lambda_{\text{max.}}$	0·12000	0·11996	0·10659	0·06466	0·05596	0·05450

When the angle of exit i deviates from 90° by 2° or 3°, it is necessary to introduce a small correction to $d\lambda_{\text{max.}}$, obtained for $i=90°$. If the angle of exit cannot be measured directly, it is approximately calculated by using (3).

$$\text{Sin } 2i = -\frac{h_0\lambda^2}{2t^2}\cdot\frac{\Delta h}{\Delta i} \quad . \; . \; . \; . \; . \; . \quad (3')$$

It is generally sufficient to take $\Delta h=10$ or 15, find the corresponding Δi, and then calculate the correction. This is easily done on a photograph.

In using the plate and the echelon gratings care was taken to protect them from changes of temperature by enclosing them in wooden boxes which were lined thickly with cork plate. Nobody remained in the room while the photographic plate was exposed.

The lamp used in the present experiment was mostly of Arons-Lummer type, fed by direct current of 10 amperes at 30 volts, and cooled by water current. Sometimes a Heraeus quartz lamp was used, but generally the lines were more distinct with the Lummer lamp, especially when it was well cooled and placed under low voltage. The photographs as well as visual observation show that the lines are better defined when the line of sight is parallel to the arc than when it is transverse. This is well exemplified in the satellite next to the principal line of 5,461 (Fig. 17).

In the present investigation we did not enter into experiment on the changes in lines caused by introducing different gases into the lamp, as was recently done by Wendt.*

The plate was provided with a right-angled prism whose section is an isosceles triangle, as in the original form used by

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* Wendt, "Ann. d. Phys.," **37**, p. 535, 1912.

Lummer and Gehrcke; but it was found more convenient to change it into another, whose section is a right-angled triangle with one angle of about 22·5°, so that the plate can be used *à vision directe.*

The resolving power of the plate and of the echelon grating was nearly the same, and for wave-length of $0{\cdot}5\mu$ it is 435,000 for the echelon grating and 400,000 for the plate.

§ 4. *Arrangement.*—In order to eliminate the ghosts, which inevitably appear in echelon gratings, the spectrum was crossed by the Lummer plate after the method of Gehrcke and v. Baeyer. For this purpose the plate was placed horizontally and the echelon grating in such a position that the spectrum lines were vertical. The arrangement is shown in Fig. 1.

The light emerging from the echelon grating was made parallel by means of Zeiss micro-planars (focal lengths 7 cm., 5 cm., 3·5 cm.), and made to fall on the vertical face of the prism attached to the Lummer plate. The interference points were photographed by means of Goerz anastigmat (focal length 21 cm., aperture 3 cm.), or by photographic lenses of 10 cm. aperture, having focal lengths of 70 cm. and 123 cm. respectively.

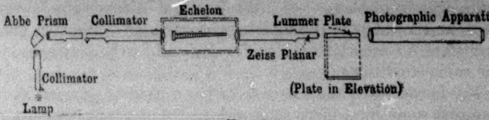

FIG. 1.

These micro-planars and photographic lenses were chosen so as to meet the special object of each experiment.

In order to obtain the photographs for the measurement of positions of satellites, and for testing the position of ghosts, lenses of long foci were generally used; while for the purpose of examining the number of satellites and for photographing the lines with short exposure, a micro-planar of long focus ($F=7{\cdot}0$) was combined with the Goerz anastigmat.

On this account the photographs given in this Paper are of various magnifications.

The Goerz lens was attached to a photo-theodolite by Günther and Tegetmeyer, as there was advantage in finding the angle of exit directly from the readings of the divided circle. In order to find the interference points of faint lines it was necessary to use lenses of short focal length, but for the determination of the deviation of satellites from the principal line, photographs obtained by lenses of longer focal lengths were used to secure more accuracy without increasing the magnifying power of the micrometer with which the relative position of the points was measured. For the latter purpose the slit in front of the echelon grating was generally opened to a width of about 0·04 mm., so that the crossed images appeared as short lines instead of points on the photographic plate. In this manner the lines of satellites could be placed between two spider lines of the micrometer, and its relative position exactly determined. When the points were crowded it was sometimes necessary to work with a slit only 0·01 mm. wide. In spite of the small quantity of light which was able to pass through the echelon as well as the plate, the exposure did not exceed five or six hours even in the very insensitive part of the spectrum.

For the photography of yellow and green lines, "panchromatic spectrum plates" of "Wratten and Wainwright" were used, and for lines in the violet region we used "Wratten process plate." Care was taken to develop and fix the plates under the same conditions.

The focussing of the lens was rather a tedious process, as it required sometimes more than a dozen photographs to obtain a sharp focus, especially for lines in the almost invisible violet region. But in such a region of the spectrum exposure did not last more than 10 or 20 minutes before we can obtain a fairly good image. Thus we can utilise the bright image given by the echelon in shortening the time of exposure of photographic plates.

§ 5. *Crossed Spectra.*—We tried different methods of crossing the spectra. The echelon spectrum was crossed with that obtained by a metallic plane grating ruled on a Rowland engine; but, owing to the faintness of light and the low resolving power of the grating, the results were by no means comparable with those obtained by the combination of the echelon and the Lummer plate. The same remark applies to the crossed spectra of two echelon gratings. The echelon above described was crossed with another belonging to the Tōkyō Higher Normal School, having a resolving power of 140,000.* As it was necessary to work with very small slits, the quantity of light was generally insufficient to show the details of the spectra, though it was much superior to the crossed spectra of echelon and ordinary grating.

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* The instrument was placed at our disposal through the courtesy of Prof. Noda, to whom our best thanks are due.

§ 6. *Advantages of Crossing the Spectra.*—In all of our observations the angle of exit was nearly equal to 90°, so that by putting

$$i=90^\circ-a,$$

where a is small, the equation giving the order of spectrum becomes

$$h^2\lambda^2=4t^2\{(\mu^2-1)+a^2\},$$

neglecting higher powers of a.

Putting $\lambda=\lambda_0+\delta\lambda$, where λ_0 is the wave-length of the reference line and $\delta\lambda$ the deviation of the wave-length of satellites from it,

$$2h\lambda_0\delta\lambda=\{8t^2(\mu^2-1)-h^2\lambda_0{}^2\}+a^2.$$

In the crossed spectra given by the echelon grating and Lummer plate we may conveniently take for abscissæ the distances of the satellites from the principal line on the echelon spectrum, and hence proportional to $\delta\lambda$, and ordinates as proportional to deviation a from grazing exit.

Consequently the locus of the interference points for the same value of h and for different satellites must lie on a parabola given by the equation of the form

$$y^2-a^2=bx,$$

where a, b are constants depending on h. For consecutive values of h the parabolas cut the x axis at nearly equidistant intervals.

Thus, by tracing the curve, we are able to arrange the interference points according to different values of h. This method is sometimes of great assistance in discriminating the position of satellites, especially when they are crowded together, as observed with the echelon or the plate only. In all of our measurements we plotted the interference points from the readings of micrometer with respect to x and y axes, in order to fix the order of the spectrum.

Figs. 8, 15, 24 show at a glance the efficacy of the method, especially for 5,790 and 4,359, in which points belonging to different orders of the spectrum are mixed together.

In order to evaluate the relative positions of the satellites it is necessary to refer them to a line which is sharply defined.

It is customary to refer them to the principal line, which is generally broad and has hazy boundaries. It is, therefore, inaccurate to take the principal line as the line of reference, and the discrepancies among different observers seem sometimes to be attributable to the uncertainty in the position of the principal line. We have in most cases used a well-defined satellite as the reference line (indicated by an asterisk in the tables), and afterwards reduced it to the principal line.

With the plate, numerous orders of spectra can be measured with a micrometer, while only a single spectrum can be placed under test with the echelon spectroscope. Moreover, the optical errors are simpler in the plate. Consequently, the result is far more accurate than that with the echelon. In all the calculations which will be made hereafter we only take the values obtained by the plate into account.

§ 7. *Interference Points.*—Owing to the limit in the resolving power of the optical instruments, the fine structure of some lines cannot be exactly known. Although some lines cannot be separated, we find that interference points due to the combination of the echelon grating with Lummer plate sometimes present a singular appearance. When the point is due to the intersection of simple strong lines it has a head and a tail, if we may so call the tapered ends of the point, which are in the direction of the tangent to the parabola joining the interference points of the same order. This is almost always exemplified in the photographs of the crossed spectra. With some points, however, the appearance is greatly modified. The head and tail, instead of being simple, sometimes present a complicated appearance. This is observed also when there is a weak satellite near a strong line, especially when the photograph is of long exposure. By this analogy we may, perhaps, be able to draw some inference as to whether the line is simple or not from the appearance of the points, although the lines may not be distinctly separated. The same remark applies to the examination of the blackened line of photographic plates by a thermopile (§ 8).

When there is a ghost in the neighbourhood of a strong line, the interference points taper towards the ghost in the direction of the x axis in the diagram of interference points when it is due to the echelon, and in the y direction when it is attributable to the plate. An example of the former case is seen in the crossed spectra of 4,078 (Fig. 29, Plate VII.); faint traces of the ghosts due to the plate were only noticed in the line 5,461. The plate was, therefore, nearly free from ghosts. A number of ghosts were found in the echelon spectrum of the lines 5,461, 4,359 and 4,047, which may be seen in the photographs given in Figs. 16, 27 and 33 (Plates III., VI. and VIII.).

§ 8. *Relative Intensity.*—In spite of the numerous investigations on the position of satellites of mercury lines, very little has been done on the relative intensities of the satellites. The exact photometry of these lines would be a matter of great difficulty, but as the deviations of satellites are only a fraction of Ångstrom unit in most cases, we may treat it as monochromatic, and hence assume the constant of the Schwarzschild's* law of blackening of photographic plates to be the same for the principal line and the satellites. Under this assumption, we may measure the relative intensities when the blackenings of the lines to be compared lie within the normal regions of the curves of blackening (Schwärzungskurve).

For this purpose, it was, in the first place, necessary to photograph the echelon spectrum under several different degrees of exposures. Firstly, normal exposure for the principal line and strong satellites, for which the faint lines are mostly in the condition of under-exposure; secondly, normal exposure for satellites of strong and mean intensity, by which the principal line is over-exposed; thirdly, normal exposure for fainter lines, and so on. By the combination of these plates we may arrive at an approximate value of the relative intensities by successive comparisons. The echelon spectra of two successive orders were generally photographed in the position of minimum deviation, by which the satellites were all seen between the images of the principal line, which were equally intense on the plate.

In order to secure the constancy of the current and voltage, the Heraeus mercury lamp was found preferable to an Arons-Lummer lamp. Both lamps were used in the present experiment. For the measurement of the relative intensity the spectrum of the echelon grating was photographed without the interposition of the Lummer plate. The space between the consecutive orders of the spectrum ranged from 3 mm. to 9 mm. on the photographic plate. This plate was attached to a micrometer and placed almost in contact with a slit (17 mm.

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* Schwarzschild, "Publik. d. Kuffner. Sternwarte," 5, 1900.

long, 0·05mm. wide), so that the lines were parallel to it. The arrangement is shown in the following figure (Fig. 2).

The light from a Nernst lamp (110 volt, 1 ampere) was made parallel by a quartz lens and passed the plate at normal incidence. The current feeding the lamp was carefully adjusted during the observation. A Rubens' thermopile, consisting of 24 junctions of iron and constantan wire, was placed behind the slit, and the light passing through the slit fell on the junctions, which were arranged in a straight line. The deflection of a D'Arsonval galvanometer, connected with the pile, was read when a successive displacement of 0·05 mm. was given to the plate.

There was great difficulty in keeping the zero of the galvanometer unchanged, as the metal case containing the thermopile got gradually heated, so that the temperature of the cold junction was gradually rising. To avoid this inconvenience, the metal case of the pile was closed in the front by a window of thin microscope cover glass, and was placed within a wooden box filled with cotton. A slow current of *dry* and dust-free air, whose temperature was made constant by passing it through a long lead tube immersed in a large water tank, was maintained through the metal case by a water pump, the pressure difference being 3 mm. of water. By this means the zero of the galvanometer remained nearly constant. Of course, it was necessary to pass the air current for several hours before starting the observation.

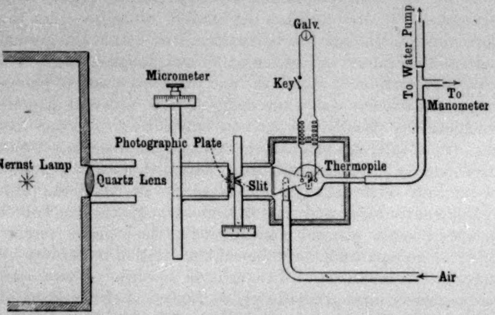

FIG. 2.

Usually it took seven or eight hours to examine the satellites of a single line.

Objections may be raised against this method of measuring the relative intensities by the blackening of the photographic plate. Fortunately the deviations of satellites are all very small, so that the question as to the validity of assuming the constant of Schwarzschild's law to be the same for different wave-lengths does not come into the problem now under discussion. The effect of diffraction and scattering due to the line as well as the slit was entirely neglected. This may not be negligibly small. The absorption of light by the photographic plate and the film, as well as the sensibility of the film, are considered to be uniform throughout the length of the plate measured. These are some of the defects attached to the above method, but to the first approximation the galvanometer indications may doubtless be taken as a measure of relative intensity. The accompanying figure shows how the

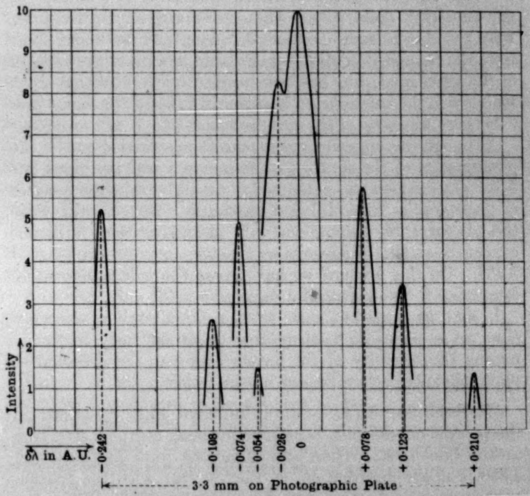

FIG. 3.—INTENSITIES OF THE SATELLITES OF λ = 5,461.

galvanometer is affected as the plate is made to pass over the slit. We shall afterwards state that the intensities in some groups of satellites are regularly distributed.

The existence of the ghosts in the photographs taken with the echelon grating alone causes disturbance in the blackening due to the real line, especially when the ghost is very near to the line. In addition to this, when a feeble line is in the neighbourhood of a strong one, the former is almost masked by the latter, but we occasionally notice a slight protuberance which disturbs the smoothness of the curves traced by plotting the galvanometer deflections. That these slight protuberances were not due to some accidental causes, such as might come from the change of intensity of the lamp, or that of the velocity of the air current, &c., has been verified by the fact that when we moved the micrometer backwards exactly the same protuberance was seen in the reverse way.

Although the intensity measurement was carried out on the photographs of the six lines of mercury, it was the green line 5,461 alone which was examined in detail by taking a number of photographs under different exposures. For the other lines the photographs were taken only in two or three different exposures, so that the results were not so accurate as in that of 5,461. On this account we have given the values of relative intensities to two places of decimals for 5,461 only.

It may be remarked that, when a faint satellite was very close to a strong line, there was much difficulty in determining the intensity of the former owing to the diffusion of photo-chemical action caused by the latter.

§ 9. *Satellites.* 5,790.—Of the different lines examined in the present experiment, none presents such complex structure as the line 5,790. There are numerous lines in 5,461 and 4,359, but the distribution, as they appear with the echelon only or crossed by the plate, is tolerably simple. With the yellow line 5,790 the complexity is due to the existence of two satellites, both at about −1·0 Å.U. from the principal line. The stronger line of the two was noticed by several previous investigators, and both lines were directly photographed with a Michelson grating by Gale and Lemon.* In the course of our investigation the stronger satellite was easily photographed by a Rowland concave grating (radius 10½ ft., 14,438 lines to the inch), of which Fig. 5 shows its position with respect to the principal 5,790 and the neighbouring line 5,769, already in the first order spectrum. The distance of the satellite from the principal is about one-twentieth of the distance between the two strong lines. This line and its companion appear in the spectrum of the echelon grating mixed with other satellites (Fig. 6), which lie very near together, and are of the same order as the principal, while the said satellite is two orders higher. To discriminate the lines from other faint lines it is necessary to cross the echelon with the plate (Fig. 7). By measuring the position of the interference points by means of a micrometer, and joining the points belonging to different satellites by parabolas, which form loci of points for the same orders of spectra, we easily find that there are distinctly two series of points which belong to different orders of spectra from other neighbouring points. The result of micrometric measurements is given in Fig. 8, and the copy of the original in Fig. 7 (Plate I.). The appearance of the echelon spectrum with the respective positions of the other satellites are given below the diagram of the interference points. The ghost is marked with the letter G. They are all blended together, so that it seems impossible to discriminate the distribution of satellites with the echelon grating alone, as the accompanying diagram of the echelon spectrum will show (Fig. 8).

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* Gale and Lemon, "Astrophys. Jour.," **31**, p. 78, 1910.

Although the crossed spectra show beyond doubt the positions of these satellites, we tried to bring fresh evidence as regards the positions of the satellites in the echelon spectrum.

Instead of using the spectrum given by the Abbe prism in the echelon apparatus as constructed by Hilger, we have projected the line given by the concave grating, in which the stronger satellite −1 Å.U. was distinctly separated from the principal line, on the slit in front of the echelon grating, so that it can be included or excluded in obtaining the echelon spectrum. The spectra, with and without the said satellites, are given in Figs. 9 (*a*) and 9 (*b*) (Plate II.) respectively. It shows that the third strong line from the left is due to the satellite about −1·0 Å.U. distant. The fourth line is a little displaced in relative position, which shows that the neighbouring satellite is almost coincident with the strong line. It will be worthy of remark that where there is some doubt as to the legitimacy of the position of satellite we may bring in the aid of the instrument of high dispersion, and by process of elimination arrive at a correct result.

A discussion was raised by Gmelin* as to the existence of the satellite $+0{\cdot}164$. By a method which is quite analogous to

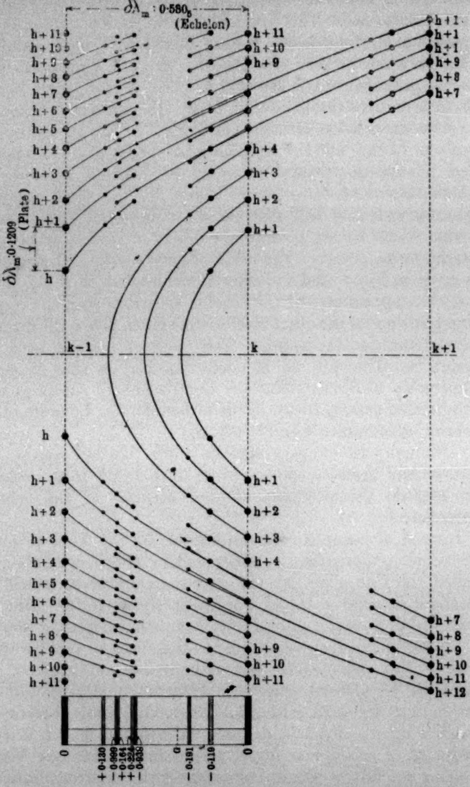
$\delta\lambda_m : 0{\cdot}580_5$
(Echelon)
$\delta\lambda_m : 0{\cdot}1209$
(Plate)
h
h+1
h+2
h+3
h+4
h+5
h+6
h+7
h+8
h+9
h+10
h+11
h+12
k−1
k
k+1
0
+0·130
−0·099
+0·164
+0·224
−0·036
0
−0·191
−0·119
0

Fig. 8.

that of crossed spectra, he showed that, in place of the said line, there is a satellite $-0{\cdot}374$. With the echelon here used, and also with the crossed plate, the line falls in the neighbourhood of −0·938 and +0·224. Of the numerous photographs which have been taken with different micro-planars and photographic lenses it was difficult to see distinctly the exact position of the said satellite. There are several indirect evidences for its existence, but we have not given it in the figure as its existence cannot be directly ascertained.

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* Gmelin, "Ann. d. Phys.," **33**, p. 17, 1910.

As to the faint line +0·164, which appears distinctly in our crossed spectra, and has been measured by Janicki,[*] Galitzin,[†] and Lunelund,[‡] there is not a least doubt of its existence. It may be due to the accidental coincidence of the line with the alleged −0·374 in the echelon spectra of the above-mentioned observers that the line was cancelled by Gmelin. It is also very curious that −0·938, which first appears in the observations of Gale and Lemon, had not been noticed previously. In our case it is altogether impossible to discriminate it from +0·224 in the echelon spectrum ; but the interposition of the Lummer plate places its existence beyond doubt, as illustrated in the diagram of the interference points. The order of plate spectrum for 0·938 is much higher than that for the neighbouring point +0·224. This is of special interest, showing how the crossed spectra can sometimes analyse the coincidence of several lines in a single echelon spectra by separating them into different interference points.

Instead of giving a table of the results of different observers, Fig. 10 will present at a glance the coincidence as well as the discrepancy of different measurements. The thickness of the line is drawn proportional to the intensity.

It is to be noticed that, the principal line being wide and diffuse, the mean point is difficult to determine ; the consequence is that the distribution of the satellites by one observer is one-sided compared to that of the other. The results of our measurement are tabulated below :—

Echelon ..	−0·996	−0·931	−0·196	−0·129	[§]0	+0·132	+0·161	+0·217
Plate......	−0·999	−0·938	−0·191	−0·119	[§]0	+0·130	+0·164	+0·224
Intensity	2	1	2	6	10	3	2	4

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[*] Janicki, "Ann. d. Phys.," **19**, p. 36, 1906.

[†] Galitzin, "Bull. de l'Acad. Imp. d. Sc. de St. Petersburg," p. 159, 1907.

[‡] Lunelund, "Ann. d. Phys.," **34**, 505, 1911.

[§] The line marked by an asterisk is the reference line in the determination of $\delta\lambda$.

We consider hereafter the result obtained by the plate as more accurate than that by an echelon, inasmuch as the optical

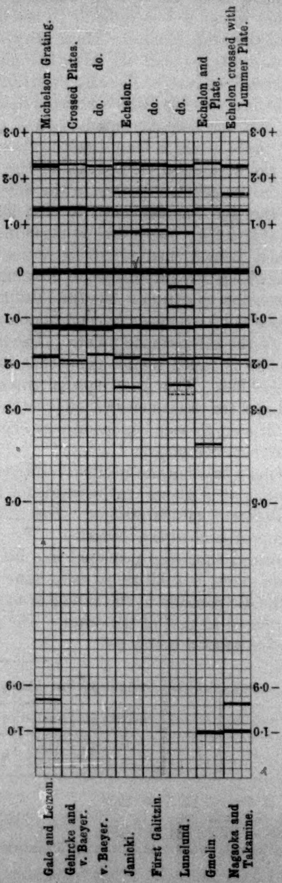

FIG. 10.—STRUCTURE OF λ = 5 790.

errors attending the former are simpler than the other, and as micrometric measurements can be made on numerous orders of spectra.

Gmelin showed that +0·084 of Janicki and +0·086 of Galitzin are coincident with the satellite −1·0 Å.U. Lunelund recognised that +0·082 of his observation would correspond to the above satellite at the position of

$$-2d\lambda_{\text{max.}}+0{\cdot}082=-1{,}006 \text{ Å.U.},$$

but he had no means of detecting the difference in order from the other satellites.

5,769.—This line is quite simple. It has three satellites, of which two are nearly symmetrical about the principal in position as well as in intensity. The third satellite seems as if it were the ghost of the principal, but micrometric measurement shows distinct deviation from the position which it must occupy if it were a false line due to echelon.

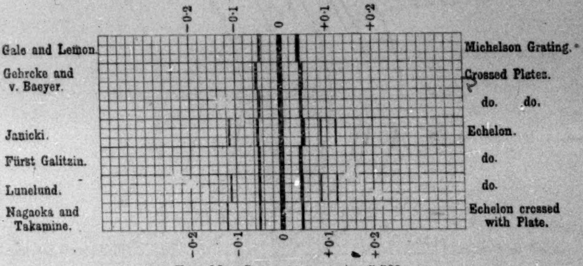

FIG. 13.—STRUCTURE OF λ=5,769.

Fig. 11 shows the crossed spectra and Fig. 12 the echelon spectrum.

Our measurements are as follows :—

Echelon............	−0·109(?)	−0·049	*0	0·046
Plate...............	−0·121	−0·050	*0	0·044
Intensity	1	3	10	3

The lines +0·084 and +0·121 given by Lunelund did not appear in the crossed spectra, and are probably ghosts. The results of various observers are shown diagrammatically in Fig. 13.

5,461.—Much interest attaches to this line, as it is the brightest and the most accurately examined of the lines of mercury. The crossed spectra (Fig. 14, Plate III.), the diagram of the interference points (Fig. 15) and the echelon spectrum (Fig. 16, Plate III.) show at once the presence of eight satellites, of which the existence can hardly be doubted. In order

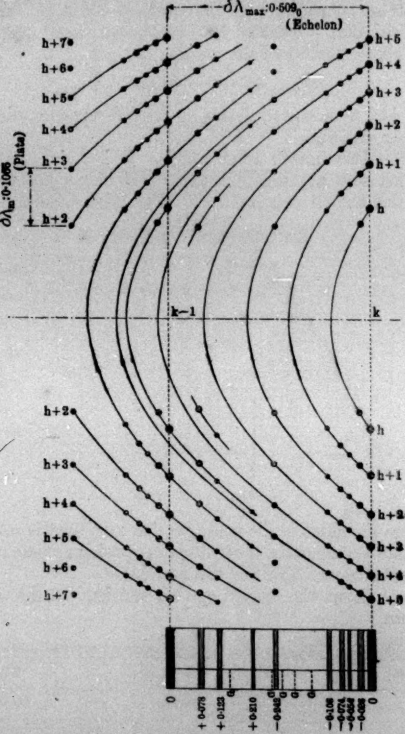
$\delta\lambda_{\max}$:0·509$_0$
(Echelon)
h+7
h+6
h+5
h+4
h+3
h+2
$\delta\lambda_m$:0·1066
(Plate)
h+5
h+4
h+3
h+2
h+1
h
k−1
k
h+2
h+3
h+4
h+5
h+6
h+7
h
h+1
h+2
h+3
h+4
h+5
0
+0·078
+0·123
G
+0·210
G
−0·242
G
G
G
−0·106
−0·074
−0·054
−0·033
0

FIG. 15.

to separate the principal line from its nearest companion, −0·026, it is necessary to run the lamp at low voltage, and place the line of sight in the direction of the arc. Also good cooling of the tube is necessary. The photographs of echelon and of crossed spectra taken under these conditions are shown in Figs. 17 and 18 (Plate IV.). By long exposure of the photographic plate we noticed fine ghosts (indicated by G in the diagram annexed to Fig. 15) in the neighbourhood of the line −0·242 and +0·210; these are very much like those observed by Stansfield, who, however, considered the real line −0·054 also as a ghost. The faint line +0·210 is not easy to measure, but its existence is beyond question (Fig. 19, Plate IV.).

The results of different observers are given in the diagram below.

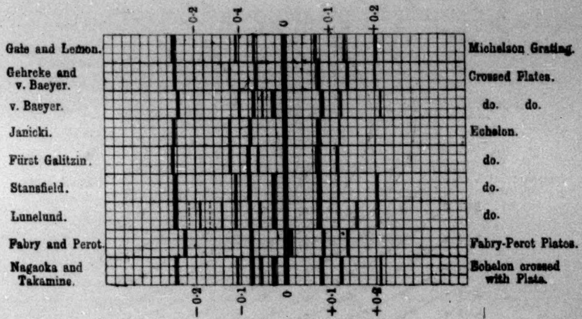
−0·2
−0·4
0
+0·1
+0·2
Gale and Lemon.
Gehrcke and v. Baeyer.
v. Baeyer.
Janicki.
Fürst Galitzin.
Stansfield.
Lunelund.
Fabry and Perot.
Nagaoka and Takamine.
Michelson Grating.
Crossed Plates.
do. do.
Echelon.
do.
do.
do.
Fabry-Perot Plates.
Echelon crossed with Plate.
−0·2
−0·1
0
+0·1
+0·2

FIG. 20.—STRUCTURE OF λ=5,461.

Our measurements are as follows :—

Echelon...	*−0·246	−0·113	−0·080	−0·055	−0·026	0	+0·072	+0·118	+0·204
Plate	*−0·242	−0·108	−0·074	−0·054	−0·026	0	+0·078	+0·123	+0·210
Intensity	5·22	2·62	4·92	1·50	8·26	10	5·77	3·47	1·34

These numbers agree very well with those found by v. Baeyer. Several of the lines cited by Lunelund occupy similar positions to the ghosts in our echelon spectrum, and will probably have no real existence.

We shall afterwards see that the intensities of the different satellites are in a regular order.

4,359.—This strong line is accompanied by 10 satellites, so that the crossed spectra are by no means simple; but as they

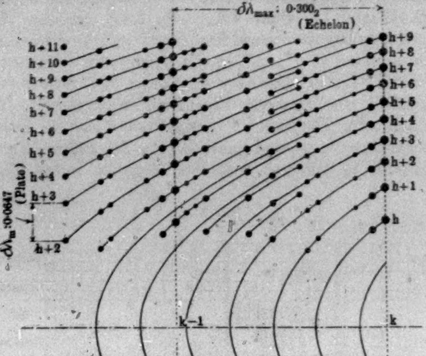
$\delta\lambda_{\max}$: 0·300$_2$
(Echelon)
$\delta\lambda_m$: 0·0847
(Plate)
h+11
h+10
h+9
h+8
h+7
h+6
h+5
h+4
h+3
h+2
h+1
h
k−1
k
0
+0·017
+0·030
+0·045
G
G
+0·206
G
−0·160
G
+0·188
−0·110
−0·097
G
−0·048
−0·023
0

FIG. 24.

are not so widely distant from the principal as in 5,790, the discrimination as to the order and position of the interference points is not so tedious as with the yellow line. The echelon spectrum is accompanied by five ghosts (G in annexed diagram of Fig. 24), which are so intense that they may be easily mistaken for real lines, had they not been eliminated by crossing with the Lummer plate. More than 100 photographs were taken before we were sure of the existence of the lines. Fig. 21 (Plate V.) shows the copy of an original, Fig. 22 enlarged, Figs. 25 and 26 (Plate VI.) more magnified in the higher orders of the crossed spectra. The photographs of the echelon spectrum, Figs. 23 and 27, indicate how misleading they are if we have only to rely on them without some means of separating the false from the true lines. The diagram below, indicating the results of different observers, shows how unreliable the echelon spectrum is alone.

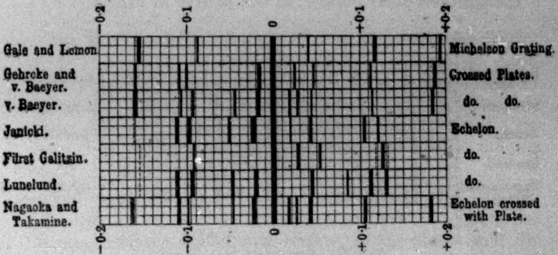
−0·2
−0·1
0
+0·1
+0·2
Gale and Lemon
Michelson Grating.
Gehrcke and v. Baeyer.
Crossed Plates.
v. Baeyer.
do. do.
Janicki.
Echelon.
Fürst Galitzin.
do.
Lunelund.
do.
Nagaoka and Takamine.
Echelon crossed with Plate.
−0·2
−0·1
0
+0·1
+0·2

FIG. 28.—STRUCTURE OF λ=4,359.

The following table gives our measurements :—

Echelon..	−0·161	*−0·108	−0·094	−0·045	−0·019	0	+0·020	+0·032	+0·045	+0·106	+0·183
Plate	−0·163	*−0·110	−0·097	−0·048	−0·023	0	+0·017	+0·026	+0·043	+0·105	+0·182
Intensity	6	4	3	1	8	10	4	1	5	5	4

The number of lines as well as the positions agree tolerably well with v. Baeyer, who had the command of the highest resolving power of all the experimenters above cited.

The diagrammatical representation (Fig. 28) of the distribution of lines shows a close resemblance with that of the satellites of 5,461.

4,078.—The structure of this line is simple, as shown in the crossed spectra (Fig. 29, Plate VII.) and in the echelon spectrum (Fig. 30, Plate VII.).

The appearance of the latter figure reminds one of the similarity with 5,790, if the line −1·0 Å.U. be eliminated. The deviations among different observers are not so large as in the case of other complex lines. Our measurements are given below :—

Echelon	−0·079	*−0·048	0	+0·029	+0·044	+0·067
Plate	−0·077	*−0·047	0	+0·032	+0·050	+0·076 4
Intensity.........	4	5	10	3	5	2

It is very singular that there is no discrepancy among different observers as to the existence of five satellites, although the positions differ slightly, as will be seen in Fig. 31.

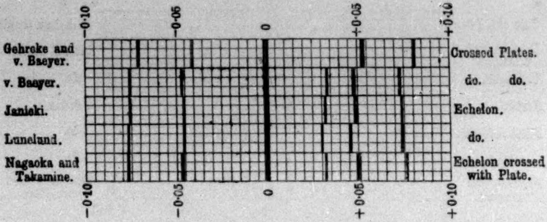
−0·10
−0·05
0
+0·05
+0·10
Gehrcke and v. Baeyer.
v. Baeyer.
Janicki.
Lunelund.
Nagaoka and Takamine.
Crossed Plates.
do. do.
Echelon.
do.
Echelon crossed with Plate.
−0·10
−0·05
0
+0·05
+0·10

FIG. 31.—STRUCTURE OF λ 4,078.

4,047.—The echelon spectrum of this line sometimes shows a number of ghosts by long exposure. Fig. 33 gives the photograph taken with 15 minutes' exposure, and Fig. 32 the

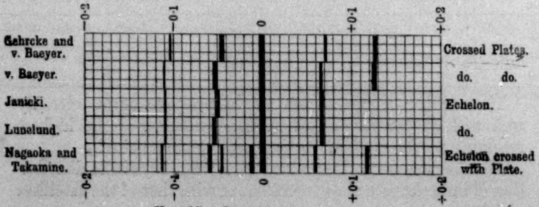
−0·2
−0·1
0
+0·1
+0·2
Gehrcke and v. Baeyer.
v. Baeyer.
Janicki.
Lunelund.
Nagaoka and Takamine.
Crossed Plates.
do. do.
Echelon.
do.
Echelon crossed with Plate.
−0·2
−0·1
0
+0·1
+0·2

FIG. 35.—STRUCTURE OF λ=4,047.

crossed spectra with a lens of short focus. A lens of long focus gives more detail (Fig. 34). The principal line is found to be double, and near it is another doublet, which has been often considered as a single broad line. The diagrammatical representation (Fig. 35) of the position of the satellites shows the discrepancies among different observers. Although we have taken photographs with exposures of several hours, we have not been able to establish the existence of the numerous satellites given by Wendt; it appears that they were mostly due to the nature of the lamp.

Our measurements are given below :—

Echelon…………	*−0·118	−0·063	−0·049	−0·013	0	+0·060	+0·117
Plate……………	*−0·114	−0·060	−0·046	−0·012	0	+0·059	+0·117
Intensity ………	3	7	4	8	10	5	5

We have further to remark that the line 4,916 is single; 4,348 and 4,339 are difficult to photograph without the intrusion of the strong line 4,359, so that it is not easy to determine how many satellites accompany them. Probably 4,348 has three, but we tried in vain to have them separated from the lines of 4,359, which inevitably appeared with 4,348 as diffused light.

§ 10. *Regularity.*—One* of us has already pointed out that the distribution of the satellites is not altogether irregular. It was, however, stated with reserve, as the ghosts were not

—	No. of Satellite.	δλ (observed).
Line 5,461 ………	(−5)	−242·2 = −(+2)×2+2·6
	(−4)	−108·3 = (−1)×4+2·3
	(−3)	− 75·0 = −(+1) +2·0
	(−2)	− 55·8 = (−1)×2−2·8
	(−1)	− 26·5
	(+1)	+ 77·0 = −(−1)×3−1·5
	(+2)	+122·4
	(+3)	+209·6 = −(−4)×2+7·0

δλ's are expressed in m.Å.U.

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* Nagaoka, "Phys. Zeit.," **10**, p. 609, 1909.

eliminated and were misleading. In the present experiments the ghosts are eliminated, so that we may enter with confidence into the discussion as regards the distribution.

It has already been stated that the $\delta\lambda$ of some satellites are in a simple ratio; a good example is afforded by the satellites of the lines 5,461, 4,078 and 4,047.

Distinguishing the satellites by the number, we find that $\delta\lambda$'s can be expressed as multiples, or sometimes as combinations of $\delta\lambda$ of other satellites as given in the third column of the lower table on previous page.

We obtain the approximate relation

$$\begin{aligned}&(-1):(-2):(-3) \text{ or } (+1):(-4)\\ &\qquad = 26{\cdot}5 : 55{\cdot}8 : 75{\cdot}0 \text{ (or } 77{\cdot}0) : 108{\cdot}3\\ &\qquad\quad\;\; 1 \;\;\; : \;\; 2 \;\; : \;\; 3 \qquad\qquad\quad\; : \;\; 4.\end{aligned}$$

The ratio is not exact, but it does not seem to be a mere chance coincidence.

——	No. of Satellite.	δλ (observed).		
Line 4,078	(−2)	−76·9	=−(+3)	−0·7
	(−1)	−46·9	=−(+2)	+3·1
	(+1)	+32·0	=−(−2)+(−1)	+2·0
	(+2)	+50·0	=−(−1)	+3·1
	(+3)	+76·2	=−(−2)	−0·7

Lines (−2) and (+3) are symmetrical about the principal; (−1) and (+2) are nearly so; (+1) is given by the difference between (−2) and (−1).

——	No. of Satellite.	δλ (observed).		
Line 4,047	(−4)	−114·1	=−(+2)	+2·6
	(−3)	− 60·1	=−(+1)	−1·6
	(−2)	− 46·4	=−(+1)−(−1)	−0·3
	(−1)	− 12·4	=−(+1)−(−2)	−0·3
	(+1)	+ 58·5	=−(−3)	−1·6
	(+2)	+116·7	=−(−4)	+2·6

The approximate symmetry between (-4) and $(+2)$, (-3) and $(+2)$ is at once evident, and $(+2)=2\times(+1)$.

If we investigate the structure of other lines we may, perhaps, find a similar relation; we believe that such a relation is not peculiar to mercury lines only, but is to be found in lines of other elements, of which manganese, investigated by Janicki, is one instance.* Examples of the symmetrical positions of the satellites with respect to the principal line are to be seen in Figs. 10, 13, 20, 28, 31 and 35. This symmetry of position is not, however, always attended with that of intensity.

In addition to this, the character of the distribution of lines is similar, especially in 5,461 and 4,359, which belong to the second subordinate series. In both of these lines the principal line is accompanied by a strong satellite on the side towards the violet, which is very near it. For these two satellites the difference in frequency from their respective principal lines is nearly the same, which is also characteristic of some alkaline elements.

Thus for 5,461 : $\frac{\delta\lambda}{\lambda^2}=88\times 10^{-11}$,

4,359 : ,, $=89$,,

4,047 : ,, $=76$,,

The last line, 4,047, shows some deviation, but as the position of the neighbouring satellite is so near that it is hardly resolved by the instruments at our command, we must wait for a more accurate determination with instruments of greater resolution.

Another characteristic is that the numbers $\frac{\delta\lambda}{\lambda}$ have values common to several of the satellites in different lines, and are multiples of those preceding them, so that some numbers occur oftener than others, if we construct a table of $\frac{\delta\lambda}{\lambda}$ for different lines.

The following table† covers all the values of $\frac{\delta\lambda}{\lambda}$ for satellites found in the present experiment. For 4,348 the mean of different observers were assumed.

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* Janicki, "Ann. d. Phys.," **29**, p. 833, 1909.

† In constructing the above table only $\delta\lambda$'s found from the plate were used; sometimes the fourth decimal was taken into account.

λ.	$\frac{\delta\lambda}{\lambda}\times 10^7$.
4,047	31
4,359	40
5,461 4,359	−49 −53
4,359 4,047	60 2×31=62
5,769 4,078 4,359	77 78 2×40=80
5,769	−87
5,461 4,359 5,461 4,359	102 98 2×49=− 98 2×53=−106
4,359 4,348	110 −110
4,348 4,078 4,047	115 −115 −115
4,078 4,359 4,359 4,047	122 2×60=120 3×40=120 4×31=124
5,461 5,461 5,461 4,047 4,047	−138 141 −3×49=−147 145 −149
4,348 4,078 4,078	186 187 −188
5,461 5,461 5,461 4,359	−198 −4× 49=−196 2×102=−204 2× 98=196
5,790 5,769	−206 −210
5,790 5,461 4,359 4,359 4,348	223 224 −222 2×110=−220 2×110=−220
4,359 4,359	240 4× 60=240
4,359	−253
5,790 4,047 4,047 4,047	283 −282 289 2×145=290
5,790 4,359 4,358	−330 3×110=330 3×110=330
4,359 5,461 5,790 4,348 4,078	−374 383 386 2×186=372 2×187=374
4,359 5,790 5,769 5,461	416 2×206=−412 2×210=−420 3×138=−414
5,461 5,461 4,359 4,359 4,348	444 2×224=448 2×222=−444 −4×110=−440 −4×110=−440
5,790	−1,620
5,790	−1,725

It will be seen that some of these numbers are remarkably coincident in several lines, and some of the numbers are multiples of others.

Interpreted in the light of Doppler's principle, the quantity $\pm\frac{\delta\lambda\,.\,c}{2\lambda}$ gives the velocity u of the approach or recession, c denoting the velocity of light. The numbers above obtained make u range from several hundred metres to many thousand metres per second. But this is simply a suggestion, and we do not mean that the above quantity is due to Doppler effect. The multiplicity of the above numbers may probably be due to the number of electronic charges which form the centres of light vibration. It would, however, be premature to speculate upon any theory which will explain the facts here described.

It may be suggested that the change of frequency, which is proportional to $\frac{\delta\lambda}{\lambda^2}$, will have the same property as $\frac{\delta\lambda}{\lambda}$; a table of $\frac{\delta\lambda}{\lambda^2}$ was therefore constructed, but no inference could be drawn from it, except the one already mentioned with respect to the companion of the principal line.

The intensities of many satellites seem to follow a simple law. A glance at Fig. 3, representing the intensities of green lines referred to the principal line, will show that they decrease proportionally to $\delta\lambda$ from the principal line for the satellites −0·026, −0·074, −0·108 on the negative side, and for +0·078 and +0·123 on the positive side. Again, the vertices of the intensity curves for −0·242, −0·108, −0·054 lie on a straight line, which makes us doubt if these lines are not the satellites of the strong line −0·242. Only the satellite +0·210 does not fit in these lines (Fig. 36, p. 28).

It must not, however, be forgotten that the ghosts in an echelon spectrum disturb the intensity of the neighbouring line to some extent.

The law of the proportionality of intensity to $\delta\lambda$ is also indicated in the intensity diagram for 4,359, 4,078 and 4,047; due to the complex structure of 5,790, it is difficult to draw any inference, as also for 5,769, in which the satellites are too few in number.

With the line 4,359 the intensity curve for −0·023 and −0·097 lies on a line with the principal, and also for −0·163, −0·110 and −0·048. Perhaps the latter lines are satellites of −0·163. On the positive side, +0·017 and + 0·026 lie on a line with the principal.

With the line 4,078, +0·050 and +0·076 form one line with the principal, while the last line may also be grouped with −0·047 and +0·032. With 4,047, −only 0·012 and −0·046 form a group with the principal line.

We have further to remark that in all of these lines two satellites of almost equal intensity, which is nearly half of that of the corresponding principal line, are always to be found—*i.e.*, −0·242 and −0·074 in 5,461, +0·043 and +0·105 in 4,359, −0·047 and +0·050 in 4,078, and +0·059 and +0·117 in 4,047.

So far as we are aware, these relations as regards the positions and the intensities of the satellites are noticed for the first time.

It will be interesting to measure the intensity with different apparatus, as the results here obtained may have been affected by instrumental errors.

At the present stage we do not wish to advance any hypothesis explaining the distribution of intensity among the satellites, but simply suggest that it will have important bearing in explaining the structure of complex spectrum lines, especially from the standpoint of the theory of ionic collisions.

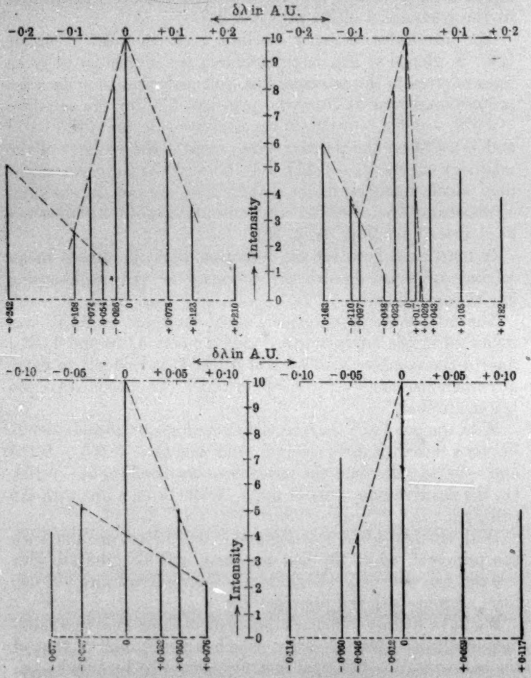

Fig. 36.—Relative Intensities of the Satellites of the Lines
λ=5,461. λ=4,359.
λ=4,078. λ=4,047.

It would also be interesting to see if these relations hold also for satellites of different lines of metals, as bismuth, cadmium, lead and others.

§ 11. *Conclusion.*—From the results of former observers, and from the present experiments, we see that the echelon spectrum is almost invariably accompanied by ghosts, and is ambiguous as regards the order of spectrum, when the satellites are not very near the principal.

These defects may be avoided, and the brightness of echelon spectrum combined with its high resolving power utilised, by crossing it with Lummer-Gehrcke plate, following the method initiated by Gehrcke and v. Baeyer.

The positions and intensities of the satellites of mercury lines are not entirely irregular, so that they seem to have some definite structure if properly arranged.

It is extremely desirable that similar researches should be extended to the spectrum lines of other heavy metals, and that the conditions under which the satellites appear or disappear should be fully investigated.

ABSTRACT.

The authors have photographed the principal lines of mercury, using an echelon spectroscope crossed by a Lummer-Gehrcke plate. They find that the 5,790 line consists of 8, the 5,769 line of 4, the 5,461 of 9, the 4,359 of 11, the 4,078 of 6 and the 4,047 of 7 components, whose positions in general agree with those found by recent observers. They point out a simple relation between the distances of the components from the principal line in each case, and a further relation between the quotient of each of these distances by the wave-length of the principal line, which holds for all the lines. The relative intensities of the component lines were determined by interposing an echelon photograph between a constant source of light and a linear thermopile, and noting the changes in the deflection of a galvanometer in series with the pile as the plate was moved across the face of the pile. In every case there appears to be a simple relation between the position and intensity of each component line.

DISCUSSION.

Prof. C. H. **Lees** pointed out that the ambiguity as to which order of spectrum any satellite belonged, could be very easily determined by transmitting the light from the echelon through a prism so as to increase or decrease the dispersion slightly, which would cause the satellites to open out from, or approach the principal line they belonged to proportionately, while the distance between the same line in adjacent orders would of course not be altered.

Prof. **Stansfield** was very much interested in the valuable work the authors had carried out, and objected only to their reflections on the character of the echelon spectroscope. He agreed with Prof. Lees that the ambiguity as to the order of spectrum lines at some distance from the principal line, mentioned by the authors, could in practice be readily avoided by employing a prism to increase or decrease slightly the echelon dispersion. The echelon he had employed showed some of its secondary diffraction maxima, as any diffraction grating approaching to perfection in its optical behaviour was bound to do. These secondary maxima were abnormally bright on one side of the principal maximum and very faint on the other. In the Paper by Stansfield and Walmsley, referred to by the authors, this had been shown to be due to a cubic aberration produced by the one-sided clamping of the glass plates. In spite of this want of symmetry, however, he thought it was only fair to the instrument to call them secondary maxima, and he hoped that the authors of the Paper would say whether the faint lines they referred to as ghosts were also secondary maxima. His own list of components for the green line only differed from that given by von Baeyer and the authors of the present Paper in the omission of the faint line at $-54m$-A., and this did not represent any difference of opinion as several of his photographs, including the one reproduced in his Paper, gave some evidence of this line. A secondary diffraction maximum which happened to come in that position was so bright that the presence of a faint primary assisting it was strongly suspected. Fabry and Perot's early values for the green line, which differed considerably from the others quoted in the Paper, were not regarded by their authors as accurate determinations. He was not aware that they had ever published them except in correspondence with Prof. Zeeman. He considered that the agreement between the results obtained by widely differing methods was fairly satisfactory. There was another effect which should not be overlooked when working with the echelon which was due to the light reflected forwards and backwards between the plates. This produced Fabry and Perot bands superposed on the ordinary spectrum. These were chiefly visible on broad lines, which thereby looked ribbed, and caused an erroneous splitting up of the lines. This could be detected by rotating the echelon slowly, when the Fabry and Perot bands will move fastest, and will, therefore, move over the surface of the line they are situated on.

Thanks were returned by the meeting to Prof. H. Nagaoka for having communicated his valuable Paper to the Society.

PLATE I.

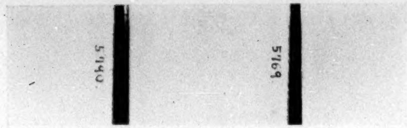

Fig. 5.—Magnification: 8.

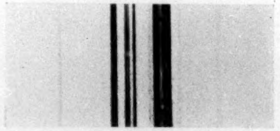

Fig. 6.—Magnification: 7.

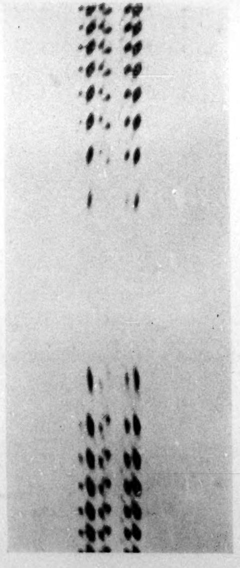

Fig. 7.—Magnification: 10.

To face page 30.]

PLATE II.

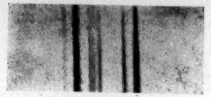

Fig. 9a.

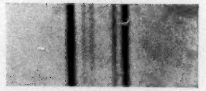

Fig. 9b.—Magnification: 12.

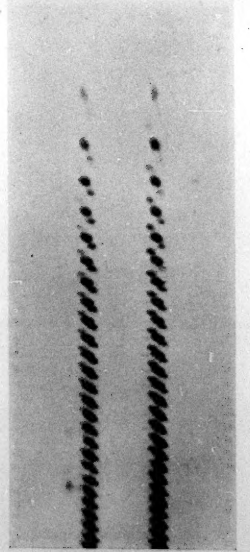

Fig. 11.—Magnification: 12.

PLATE III.

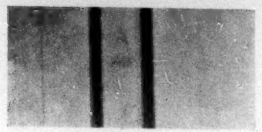

Fig. 12.—Magnification: 7.

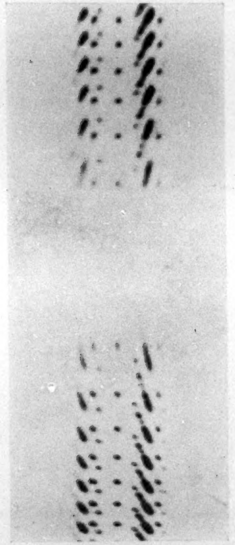

Fig. 14.—Magnification: 12.

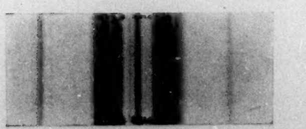

Fig. 16.—Magnification: 8.

PLATE IV.

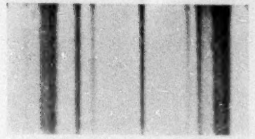

Fig. 17.—Magnification: 6.

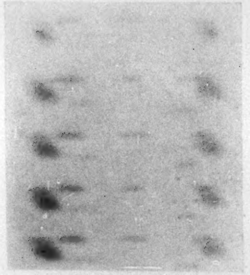

Fig. 18.—Magnification: 8.

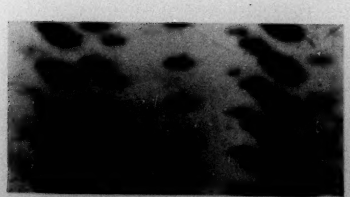

Fig. 19.—Magnification: 10.

PLATE V.

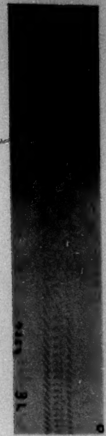

Fig. 21.—Actual Size.

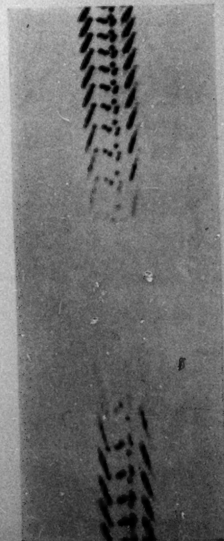

Fig. 22.—Magnification: 10.

PLATE VI.

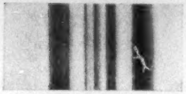

FIG. 23.—MAGNIFICATION: 6.

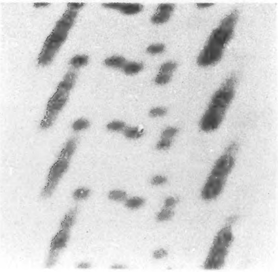

FIG. 25.—MAGNIFICATION: 8.

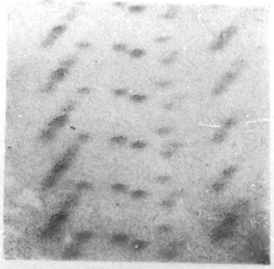

FIG. 26.—MAGNIFICATION: 8.

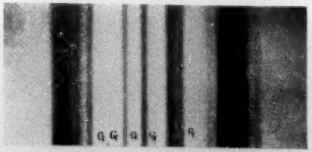

FIG. 27.—MAGNIFICATION: 4.

To face page 30.]

PLATE VII.

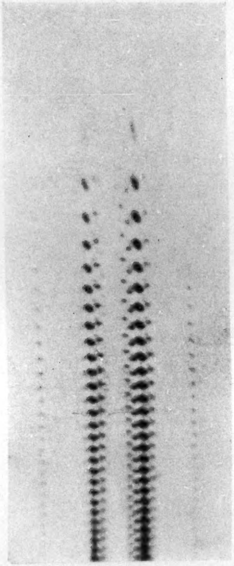

FIG. 29.—MAGNIFICATION: 10.

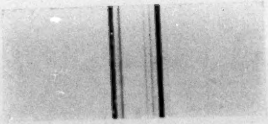

FIG. 30.—MAGNIFICATION: 10.

PLATE VIII.

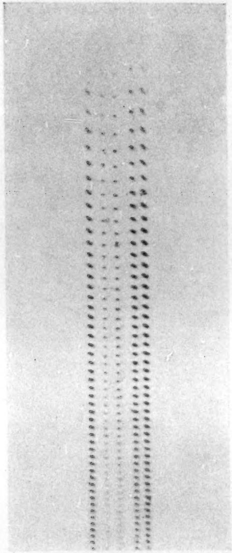

Fig. 32.—Magnification : 10.

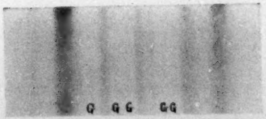

Fig. 33.—Magnification : 10.

PLATE IX.

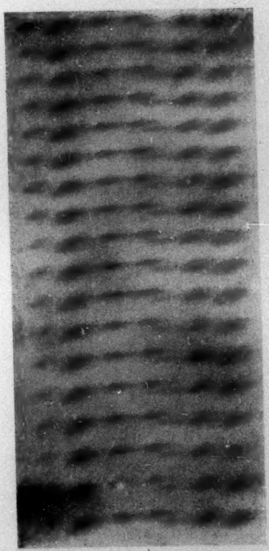

Fig. 34.—Magnification : 8.

II. *Note on the Mutual Inductance of Two Coaxial Circular Currents.* *By Prof.* H. NAGAOKA, *Imperial University, Tokyo.*

RECEIVED AUGUST 27, 1912. READ OCTOBER 25, 1912.

THE "Bulletin" of the Bureau of Standards, Vol. VIII., No. 1 contains a résumé of different formulas for the mutual inductance of two coaxial circular currents. Several tables for the numerical evaluation of mutual inductance are given, but it appears to me that nearly all practical cases can be calculated with accuracy of about one part in a million, by means of a short table, which is applicable to all cases, for which the circles may be far apart or they may be *nearly* in contact.

Denoting the radii of the circles by A and a, and the distance between the centres by b, Maxwell* gave two formulas :

$$\mathrm{M}=4\pi\sqrt{\mathrm{A}a}\left\{\left(\frac{2}{k}-k\right)\mathrm{K}-\frac{2}{k}\mathrm{E}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

in which $$k=\frac{2\sqrt{\mathrm{A}a}}{\sqrt{(\mathrm{A}+a)^2+b^2}}=\sin\gamma=\frac{\sqrt{r_1^2-r_2^2}}{r_1},$$

and $$\mathrm{M}=8\pi\frac{\sqrt{\mathrm{A}a}}{\sqrt{k_1}}(\mathrm{K}-\mathrm{E}) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

in which $$k_1=\frac{r_1-r_2}{r_1+r_2},$$

$$r_1=\sqrt{(\mathrm{A}+a)^2+b^2},$$
$$r_2=\sqrt{(\mathrm{A}-a)^2+b^2}.$$

K and E are complete elliptic integrals of the first and second kind respectively, with the corresponding modulus k or k_1.

From the general expression for the mutual inductance of two circles, or from Maxwell's first formula, we can deduce the following expression† in theta-functions :

$$\mathrm{M}=-\frac{2\sqrt{\mathrm{A}a}}{\theta_2^2(o)}\left(\frac{\theta_0''(o)}{\theta_0(o)}+\frac{\theta_3''(o)}{\theta_3(o)}\right) . \quad . \quad . \quad . \quad . \qquad (3)$$

On expansion, this becomes

$$\mathrm{M}=4\pi\sqrt{\mathrm{A}a}\{4\pi q^{\frac{3}{2}}(1+3q^4-4q^6+9q^8-+\ldots)\} . \quad . \quad . \qquad (3')$$

---

* Maxwell, "Electricity and Magnetism," 2, § 701.

† Nagaoka, "Phil. Mag.", 6, p. 19, 1903.

The above series is rapidly convergent, and is convenient for numerical calculation, as the values of

$$\log\,(1+3q^4-4q^6+9q^8-+\dots)$$

have been tabulated.* When k is near unity, we have to use q_1, which is complementary to q, and

$$M=4\pi\sqrt{Aa}\frac{1}{2(1-2q_1)^2}\left\{\text{log. nat.}\left(\frac{1}{q_1}\right)(1+8q_1-8q_1^2+\dots)-4\right\}. \quad (4)$$

Although the above series converges very rapidly, it is rather tedious to find log. nat. $\left(\frac{1}{q_1}\right)$.

In Maxwell's second formula, we have the relation between k and k_1 given by

$$k=\frac{2\sqrt{k_1}}{1+k_1}.$$

From the theory of elliptic functions, we know that for the above transformation,

$$2\frac{K'(k')}{K(k)}=\frac{K'(k'_1)}{K(k_1)},$$

so that
$$q(k)=e^{-\pi\frac{K'(k')}{K(k)}}=e^{-\frac{\pi}{2}\cdot\frac{K'(k'_1)}{K(k_1)}},$$
$$=\sqrt{q(k_1)}$$

Denoting $q(k_1)$ by h, we have

$$h=q^2(k) \quad \dots\dots\dots\dots \quad (5)$$

By direct transformation, or from Maxwell's second formula, we get

$$M=4\pi\sqrt{Aa}\frac{\theta''_0(o)}{\theta'_1(o)}, \quad \dots\dots\dots \quad (6)$$

which on expansion becomes

$$M=4\pi\sqrt{Aa}\left\{4\pi h^{\frac{3}{4}}\left(\frac{1-4h^3+9h^8-16h^{15}+25h^{24}-\dots}{1-3h^2+5h^6-7h^{12}+9h^{20}-\dots}\right)\right\}\dots \quad (6')$$

Since h is a small quantity, the fractional term is near unity; we may therefore write it $1+\varepsilon$, so that

$$M=4\pi\sqrt{Aa}\{4\pi h^{\frac{3}{4}}(1+\varepsilon)\} \quad \dots\dots\dots \quad (7)$$

On expansion

$$\varepsilon=1+3h^2-4h^3+9h^4-12h^5+\dots-$$

which is the same as that given in (3′), if we take account of the relation (5). Constructing the table of $\log_{10}(1+\varepsilon)$ for values of h from 0 to 0·2, we can easily calculate M by (7) for almost all

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* Nagaoka, "Journ." Coll. Sci., Tokyo, 27, Art. 6, 1909; "Bull." Bur. Stand., 8, p. 215-220, 1912.

Table *of* $\log_{10}(1+\epsilon)$.

h	0	Δ	1	Δ	2	Δ	3	Δ	4	Δ	5	Δ	6	Δ	7	Δ	8	Δ	9	Δ
0·00	0·0000000	13	0·0000013	39	0·0000052	65	0·0000117	91	0·0000208	116	0·0000324	141	0·0000465	167	0·0000632	193	0·0000825	218	0·0001043	243
0·01	0·0001286	268	0·0001554	293	0·0001847	317	0·0002164	343	0·0002507	367	0·0002874	392	0·0003266	416	0·0003682	440	0·0004122	465	0·0004587	489
0·02	0·0005076	513	0·0005589	537	0·0006126	561	0·0006687	584	0·0007271	608	0·0007879	632	0·0008511	656	0·0009167	678	0·0009845	702	0·0010547	726
0·03	0·0011273	748	0·0012021	772	0·0012793	794	0·0013587	818	0·0014405	840	0·0015245	863	0·0016108	885	0·0016993	908	0·0017901	930	0·0018831	953
0·04	0·0019784	976	0·0020760	997	0·0021757	1019	0·0022776	1041	0·0023817	1063	0·0024880	1086	0·0025966	1106	0·0027072	1129	0·0028201	1150	0·0029351	1171
0·05	0·0030522	1194	0·0031716	1214	0·0032930	1235	0·0034165	1258	0·0035423	1278	0·0036701	1299	0·0038000	1319	0·0039319	1341	0·0040660	1362	0·0042022	1382
0·06	0·0043404	1403	0·0044807	1423	0·0046230	1444	0·0047674	1465	0·0049139	1485	0·0050624	1505	0·0052129	1525	0·0053654	1545	0·0055199	1566	0·0056765	1585
0·07	0·0058350	1605	0·0059955	1625	0·0061580	1645	0·0063225	1665	0·0064890	1684	0·0066574	1703	0·0068277	1724	0·0070001	1742	0·0071743	1762	0·0073505	1781
0·08	0·0075286	1801	0·0077087	1819	0·0078906	1838	0·0080744	1858	0·0082602	1876	0·0084478	1896	0·0086374	1914	0·0088288	1933	0·0090221	1951	0·0092172	1971
0·09	0·0094143	1989	0·0096132	2007	0·0098139	2025	0·0100164	2044	0·0102208	2062	0·0104270	2081	0·0106351	2098	0·0108449	2117	0·0110566	2135	0·0112701	2153
0·10	0·0114854	2171	0·0117025	2188	0·0119213	2207	0·0121420	2224	0·0123644	2242	0·0125886	2259	0·0128145	2278	0·0130423	2294	0·0132717	2312	0·0135029	2330
0·11	0·0137359	2347	0·0139706	2364	0·0142070	2381	0·0144451	2399	0·0146850	2415	0·0149265	2433	0·0151698	2450	0·0154148	2467	0·0156615	2484	0·0159099	2501
0·12	0·0161600	2517	0·0164117	2535	0·0166652	2551	0·0169203	2567	0·0171770	2584	0·0174354	2601	0·0176955	2618	0·0179573	2633	0·0182206	2651	0·0184857	2667
0·13	0·0187524	2683	0·0190207	2699	0·0192906	2716	0·0195622	2731	0·0198353	2748	0·0201101	2764	0·0203865	2780	0·0206645	2796	0·0209441	2812	0·0212253	2828
0·14	0·0215081	2844	0·0217925	2860	0·0220785	2875	0·0223660	2892	0·0226552	2906	0·0229458	2923	0·0232381	2938	0·0235319	2954	0·0238273	2970	0·0241243	2985
0·15	0·0244228	3000	0·0247228	3016	0·0250244	3031	0·0253275	3046	0·0256321	3062	0·0259383	3077	0·0262460	3092	0·0265552	3108	0·0268660	3123	0·0271783	3137
0·16	0·0274920	3153	0·0278073	3168	0·0281241	3183	0·0284424	3198	0·0287622	3213	0·0290835	3228	0·0294063	3242	0·0297305	3258	0·0300563	3272	0·0303835	3287
0·17	0·0307122	3302	0·0310424	3316	0·0313740	3332	0·0317072	3346	0·0320418	3360	0·0323778	3375	0·0327153	3390	0·0330543	3404	0·0333947	3418	0·0337365	3433
0·18	0·0340798	3447	0·0344245	3462	0·0347707	3476	0·0351183	3491	0·0354674	3505	0·0358179	3519	0·0361698	3533	0·0365231	3548	0·0368779	3562	0·0372341	3576
0·19	0·0375917	3590	0·0379507	3605	0·0383112	3618	0·0386730	3633	0·0390363	3646	0·0394009	3660	0·0397669	3675	0·0401344	3689	0·0405033	3702	0·0408735	3717
0·20	0·0412452		...		...		...		...		...		...		...		...		...	

To face page 32.]

cases that occur in practice. Evidently the interval between $h=0$ and $h=0{\cdot}2$ corresponds to that between $\gamma=0°$ and $\gamma=89°\ 30'$ in Maxwell's table, which only extends from $\gamma=60°$ to $\gamma=90°$.

The accompanying table of $\log_{10}(1+\varepsilon)$ was calculated for me by Mr. T. Mishima, who made use of eight place tables of logarithms by Bauschinger and Peters.* The table can be used within a wide range, *i.e.*, when the circles are very widely apart till they are nearly in contact.

The following explanation for the use of the table will be perhaps necessary.

Calculate
$$k'_1=\frac{2\sqrt{r_1r_2}}{r_1+r_2},$$

which follows from the relation $k'^2_1=1-k_1^2$, and find
$$l=\frac{1-\sqrt{k'_1}}{1+\sqrt{k'_1}}.$$

Then
$$h=\frac{l}{2}+2\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+\ldots$$

Generally the first term is sufficiently accurate. Find $\log_{10}(1+\varepsilon)$, corresponding to the above value of h; then M will be given by (7). When great accuracy is required, it is advisable to calculate l by the formula
$$l=\frac{k_1^2}{(1+k'_1)(1+\sqrt{k'_1})^2}.$$

An example will suffice to show the use of the table.

Take $\quad A=a=25$ cm., $\quad b=1$ cm.,

then $\quad k_1=0{\cdot}9607920,\quad \frac{l}{2}=0{\cdot}1550662$

$$h=0{\cdot}1552463$$

$$\log(1+\varepsilon)=0{\cdot}0260142 \text{ by the table.}$$
$$M=1036{\cdot}666_3 \text{ cm. by (7)}$$

The last figure is, of course, uncertain, as the tables are of seven places. By using Legendre's table in Maxwell's formula (2), Rosa and Grover† give for the same case

$$M=1036{\cdot}666_1 \text{ cm.},$$

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* Bauschinger and Peters, "Logarithmisch-trigonometrische Tafeln mit acht Dezimalstellen," Leipzig, 1910.

† Rosa and Grover, "Bull." Bur. Stand., 8, p. 21, 1912.

which differs from the value found above by less than one part in a million.

If we have to use Maxwell's formula (1), and the table* compiled after it, we have to find the value of M for $\gamma=88^\circ\ 51'\ 14''$. It will be easily seen that the first and second differences are very large, so that the calculation is extremely tedious.

The chief advantage of the above table lies in the fact that nearly all cases in practice are included within a short range of argument h, and the calculation is easy to perform, as the *second* difference is generally small ($\leqq 26$).

ABSTRACT.

Methods are given for the rapid calculation of the mutual inductance of two coaxial circular currents. Maxwell's first formula is converted into theta-functions, and then expanded in a Jacobian q series. The logarithmic values of this series for various values of q have been tabulated in a previous Paper by the author. When the circles are near one another a series for M is given in terms of q_1, where q_1 is the complement of q. In this Paper the author treats Maxwell's second formula in a similar way. A table of the values of these series found, computed to six decimal figures by T. Tishima, is given. The chief advantages of this table are that nearly all practical cases are included within a short range of the argument, and the calculation is simple, as the numbers in the difference columns are small. By the help of these tables and series the mutual inductance between two coaxial currents can be easily computed to a high degree of accuracy.

DISCUSSION.

Mr. A. **Campbell** expressed great thanks for this communication. He had already used the tables given by Prof. Nagaoka in a former Paper on the subject published in Japan very largely, and those given in the present Paper covered still wider limits and enabled cases to be calculated which otherwise necessitated the use of Legrende's Tables of Elliptic Functions.

Dr. **Russell** expressed his interest in the Paper. He pointed out that the value of the mutual inductance was given to seven figures. It was certainly a step in advance to be able to evaluate so easily and to such a high degree of accuracy the simple expression for the mutual inductance. But in practice the wires used had appreciable thickness, and it was highly desirable that a more accurate formula be obtained so as to take this thickness into account.

Mr. F. E. **Smith** said he would like to express personal thanks for the Paper, and hoped that Prof. H. Nagaoka would also give tables for the mutual induction of a coil and a circle. The point raised by Dr. Russell of the thickness of the wires had been considered by Dr. G. F. C. Searle in a Paper on the Current Balance, who found that the effect was in general negligible. It was quite evident that the present Paper was both labour and time saving.

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* Maxwell, "Electricity and Magnetism," 2, 701. Also, Appendix, "Bull." Bur. Stand. 8, p. 190-192, 1912.

III. *The Absorption of Gas in Vacuum Tubes.* *By* S. E. HILL, *B.Sc.*

RECEIVED AUGUST 30, 1912. READ OCTOBER 25, 1912.

IT has long been known that the continued discharge of a current through an ordinary glass vacuum tube causes a gradual diminution in the pressure. Plücker first noticed the effect while passing the discharge by means of a Ruhmkorff's coil through a Geissler tube.*

Later on Muller and De la Rue observed the change in pressure after passing a discharge for several minutes through similar Geissler tubes, and in their later experiments they had to give up the use of tubes sealed off from the pump.†

Hutchins‡ examined the spectra of the gas enclosed in a vacuum tube, and found that if a trace of any gas be admitted into an exhausted tube and the discharge be then passed through it, the spectrum of the gas gradually disappears, leaving only the spectrum of hydrogen. This latter spectrum arises probably from the water vapour condensed on the walls of the tube, and which is so difficult to remove. Since then the knowledge of this phenomenon has been used to obtain extremely low pressures in tubes. Thus a tube run continually will reach such a degree of exhaustion that the discharge is at last unable to pass. The method has also been used to obtain "hard" rays from Röntgen ray bulbs. Many investigations have been made into the causes of this disappearance of gas, but none of the experiments performed have allowed us to decide with certainty whether the action is a definite chemical action or merely a physical absorption. J. J. Thomson inclined to the latter theory; but the experiments of R. S. Willows,§ carried out at the same time, did not bear out this idea. Willows used as his source of current a battery, as lending itself better to quantitative results than the irregular current derived from a coil. His experiments were conducted with soda, lead and Jena glass, under varying conditions of surface, &c. He came to the conclusion that most, if not all, of the gas absorbed is to be accounted for by a chemical combination with the glass. Willows also found that soda glass gave the greatest absorption, lead glass next and Jena glass least of all. He therefore recommended the use of Jena glass for vacuum tubes. Hydrogen was found to be absorbed by Jena and lead glass to a far less extent than air or nitrogen, and in the case of lead glass part of the hydrogen reappeared after resting the tube for some time.

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* Plücker, "Pogg. Annal.," 1858, Vol. CIII., p. 91.

† Phil. "Trans.," Part I., 1878, p. 155.

‡ Hutchins, "Amer. Journ. of Science," Vol. VII., 1899, p. 61.

§ Willows, "Phil. Mag.," April, 1901.

In 1907 Campbell-Swinton* published his experiments on the subject. He found that after a tube had been made to absorb a large quantity of gas, and was then heated to its fusion point, bubbles of gas appeared in the glass, which came to the surface and there burst. These bubbles appeared to be in the walls at an average depth of 0·12 mm. The glass was next crushed to powder in a special vessel, and the spectrum of the absorbed gas at once became evident. His final conclusion was that the gradual disappearance was due to the mechanical driving of the gas into the glass, and not to any chemical combination.

Soddy and Mackenzie† repeated these experiments, and obtained the Campbell-Swinton effect of the bubbles in the glass, but came to the conclusion that the gas which causes these bubbles is not the discharge gas driven into the glass. The bubbles are, in all probability, a secondary effect due to the chemical decomposition of the glass under the influence of local heating produced during the bombardment. They suggest, also, that in glass there are always present sufficient undecomposed carbonates and sulphates to account for the effect. No conclusion was arrived at as to whether the absorption of the gas was chemical or mechanical. Various other suggestions have been made. Thus, it has been suggested that the gas is shot right through the walls of the tube, but one of Willow's experiments negatives this. It has been thought that the absorption takes place at the electrodes, but this interpretation takes no note of the fact that the effect is obtained with the electrodeless discharge.

When about to commence a fresh series of experiments it seemed probable that a study of the effect with the electrodeless discharge would lead to more definite results, and this for several reasons. In the first place, any possibility of electrode action is done away with, both as regards absorption and evolution of gas. This latter proves a continual source of difficulty throughout such experiments. Secondly, with the ring discharge, the ions are not directly shot into the glass, as is the case when metal electrodes are used. The possibility of this is still less if the bulb is screened to prevent any electrostatic effects. Under these conditions, that part of the disappearance of gas due to mechanical driving into the walls of the tube will be practically eliminated or, at least, greatly reduced. The following experiments were, therefore, carried out on lines suggested by Dr. Willows. The apparatus was arranged as in Fig. 1. A is the bulb under examination, having a volume of about 200 cc. It is connected, through drying tubes, to a Toepler pump and M'Leod gauge. The gas absorption was measured by the corresponding pressure change, the pump being cut off during the readings, in order to make this greater. The secondaries of a 6 in. induction coil were connected to a spark-gap, G, consisting of two zinc balls of diameter $1\frac{1}{2}$ in. The width of the spark-gap was maintained constant throughout the experiments at about 3 mm. Two Leyden jars were connected to the spark-gap, and from these went a coil of insulated wire, of about eight turns, which was placed round the bulb A. A great increase in the potential was obtained by blowing out the spark in the gap G. This was effected by means of a motor-driven blast. A discharge, almost invisible without the blast, became brightly luminous when the spark was blown out, and also much steadier.

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* Campbell-Swinton, "Proc." Roy. Soc., Series A, Vol. LXXIX., April, 1907.

† Soddy and Mackenzie, "Proc." Roy. Soc., Series A, Vol. LXXX., February, 1908.

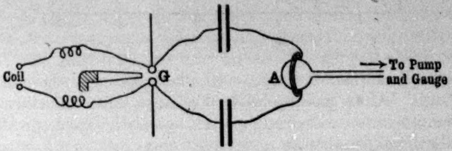

FIG. 1.

The Leyden jars, spark-gap, &c., were insulated by standing on ebonite. The method of conducting an experiment was as follows :—

The gas was pumped down to a low pressure by means of a Fleuss pump. The exhaustion was continued by the Toepler pump, testing at intervals to see if the discharge could pass. As soon as this stage was reached the pressure was noted and the spark-gap cleaned and adjusted. The discharge was then passed for a known time ($\frac{1}{4}$ hour) and the final pressure noted. The absorption could then be calculated, knowing the volume of the apparatus. The bulb was then exhausted farther and another reading taken. The series was continued until a stage was reached when the discharge would no longer pass. In no case could the discharge be made to pass at a higher pressure than about 0·4 mm., or at a lower pressure than about 0·04 mm. After passing each discharge, the difference of pressure was calculated as a percentage of the mean pressure during the experiment.

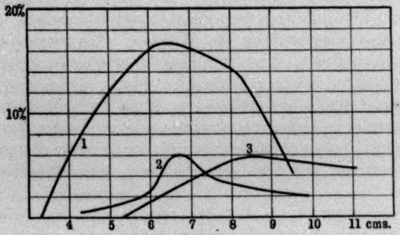

Fig. 2.—Soda Glass.

The first bulb experimented on was of soda glass. A series of readings was taken as explained above and the results plotted on a curve. Fig. 2 shows the first curve obtained.

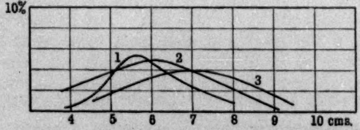

Fig. 3.—Bohemia Glass.

The ordinates are percentage absorptions, and the abscissæ the mean pressure during that absorption, given in centimetres of the gauge. The gauge read to $\frac{1}{500}$, therefore 10 cm. on the gauge represents $\frac{1}{50}$ of a centimetre actual pressure.

Some fresh air was then admitted to the bulb and allowed to stand over night to dry. Another series of readings was then taken, and after this a third series. The curves are marked respectively 1, 2, 3. It will be seen that the absorption is less each time. The general shape of the curve remains the same in each case, but the whole is, as it were, shifted bodily along to the right, towards the region of higher pressures. The greatest absorption in the first experiment was 17 per cent. and in the last only 6 per cent. It seems, then, that the glass can be "tired" by continual working. The soda bulb was then replaced by one of Bohemia glass. The absorptions obtained with this were less than half those of the soda bulb. The general shape of the curves (*see* Fig. 3) remains as before, the shift of the maximum towards the right again being noticeable.

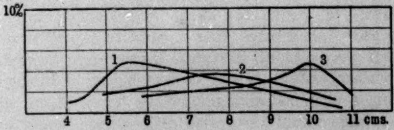

FIG. 4.—LEAD GLASS.

A lead glass bulb was next tried. The general shape of the curves was again the same, the magnitude of the effect being about the same as with Bohemia glass (Fig. 4).

The last bulb used was of Jena glass. The absorption of air was of about the same magnitude as that of the lead and Bohemia bulbs (Fig. 5).

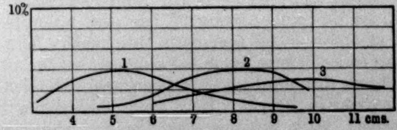

FIG. 5.—JENA GLASS.

It will be noticed that the "tiring" effect is not so great with the others as with the soda glass. The order of absorption was thus: soda, lead, Bohemia, Jena, decreasing from first to last.

The glasses were then laid aside for over two months, and the series was then repeated to see if they had recovered their power of absorption during this period. The results showed that the glass had recovered practically none of its absorptive power on standing. By the end of this second series of readings the maximum absorption for the soda glass had decreased to 2½ per cent., compared with an original 17 per cent. With the lead, Bohemia and Jena glasses similar results were obtained—*i.e.*, in each case the glass is becoming "saturated" with the gas. At this stage it was noticed that the soda and lead glasses had each a peculiar deposit on the neck of the bulb. The layer was of a dark brownish colour and its thickness very slight—of the magnitude of the wave-length of light, as shown by the colours it exhibited.

It was next decided to study the behaviour of hydrogen in the bulbs. If, during the above experiments, any chemical actions have taken place, it is natural to expect that various oxidation products of the glass have been formed. With hydrogen we should expect corresponding reducing actions. Assuming a chemical change to have taken place, we might expect a large initial absorption of hydrogen, going to reduce the oxidation products formed during the air experiments. On the other hand, if the gas is held in the glass mechanically, when an equilibrium state has been reached with air we should expect that a nearly steady state has been established for any other gas. On these grounds it ought to be possible to decide between the two theories that have been advanced to account for the absorption. The soda bulb was put on again and filled with hydrogen and pumped out several times. As soon as the discharge was started the brown layer on the neck of the bulb gradually began to disappear. The first absorption of hydrogen was enormous, the pressure falling from 14·45 cm. down to 5·05 cm. on the gauge, giving 95 per cent. absorption, calculated as above. This absorption then rapidly fell off, and after four series became much the same as that for air. The lead bulb was then filled with hydrogen, and with this the first absorption was 76 per cent. Here, again, the layer on the neck disappeared. With Bohemia glass the first absorption was 40 per cent., falling to 10 per cent. The Jena glass, which gave the smallest absorption with oxygen, gave the smallest with hydrogen, the maximum being about 8 per cent.

If, now, the hydrogen has reduced part of the oxidation products formed, it seems probable that it should have the effect of increasing the power of re-absorbing oxygen. To test this the bulbs were again filled with air, and its absorption noted. In all cases the absorption was found to have increased, although only with the Jena and Bohemia bulbs did it approach near to its original value. The curves are shown in Figs. 6 and 7. In each case curve 1 is the last series with air before hydrogen was used, and curve 2 is the series with air after hydrogen was used.

In each case the hydrogen has had the effect of increasing the subsequent absorption of oxygen.

To see if any electrostatic effects of the discharge come into play, the bulb was wrapped round with wet blotting paper and a series of absorptions taken. No difference in effects was obtained, however.

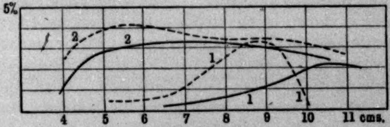

FIG. 6.
Full line Curves = Soda.
Broken line Curves = Lead.

It was also noted that some of the bulbs showed the "afterglow" effect well. It was also thought that, if there had been any bombardment of the glass by the discharge, the phenomena of "thermo-luminescence," described by Wiedmann and Schmidt,* might be obtained as in the case of bombardment by cathode rays. None of the bulbs, however, showed this effect. A review of the experiments shows that, in accordance with Willows' results, Jena glass is the best glass for vacuum tubes, as it shows least absorption. Even with soda glass the pressure variations can be reduced by saturating the glass with gas before it is sealed off from the pump. All that is required is to run the tube for some hours and admit

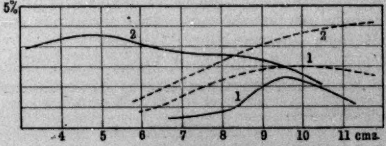

FIG. 7.
Full line Curves = Bohemia.
Broken line Curves = Jena.

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* Wiedemann and Schmidt, "Wied. Annal.," 59, 1895.

fresh gas until very little more is absorbed. Willows stated that Jena and lead glass absorb hydrogen to a less extent than air. This was found not to be the case, and his results are probably explained by the evolution of gas from his electrodes. The order of absorption with oxygen (air) gives soda, lead, Bohemia and Jena, and the same order holds for hydrogen.

To summarise: It seems most probable that the absorption is chiefly of a chemical nature. That some chemical changes are present is shown by the deposits on the necks of the bulbs. Thus, the deposit on the soda glass caused by air was removed by the hydrogen. A brown deposit on the Bohemia and lead glass likewise disappeared with hydrogen. With the Jena glass a slight colouration was given by oxygen, but with hydrogen a distinct black ring was formed which gradually diffused along the tube. Again, Strutt has shown that even nitrogen becomes unusually active in the discharge. These deposits, unfortunately, are too slight for chemical analysis, but they may be modifications of the alkalis in the glass. The absorption of the active gas hydrogen is much greater than that of oxygen, and the various glasses keep the same order of absorption with both gases.

Soddy and Mackenzie* obtained correspondingly small absorptions with the inert gases, and their experiments also tend to negative the mechanical theory of the absorption. It is, however, extremely difficult to devise a crucial experiment to definitely decide the cause of the absorption, and there is room for much more work on the subject.

In conclusion, I offer my best thanks to Dr. Willows for valuable guidance and suggestions during the experiments.

CASS INSTITUTE, E.C.

ABSTRACT.

This Paper is an account of experiments carried out to determine whether the absorption of gases caused by passing a discharge for some time through vacuum tubes is the result of a chemical action or is a mere physical absorption. In order to eliminate all electrode complications, the electrodeless discharge was used throughout. The bulbs examined were of soda, lead, Bohemia and Jena glass. The absorptions were noted at different pressures and curves plotted. Continued passage of a discharge causes a "saturation" effect in all the glasses. After two months none of the bulbs had recovered any of their absorptive power. If the action is chemical it is natural to expect various oxidation products to have been formed. Testing these bulbs with hydrogen we should expect a large initial absorption going to reduce these products. This was found to be the case for all

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* Soddy and Mackenzie, *loc. cit.*

the bulbs, the first reading for the soda glass giving 95 per cent. absorption of hydrogen. Having now reduced the oxidation products we should expect a reabsorption of oxygen under the discharge. This was also found to be the case. The series of readings show great regularity, the order of absorption for oxygen holding also for hydrogen. That chemical actions are present is shown by peculiar deposits on the necks of the bulbs, these being unfortunately too small for analysis. The inert gases show correspondingly small absorption, as shown by Soddy and Mackenzie, and therefore the conclusion is that the disappearance is not due to physical absorption, but to definite chemical action.

DISCUSSION.

Dr. R. S. **Willows** mentioned the fact that the inert gases, like argon and helium, also disappeared on running the discharge, but this might be due to their being carried down by the cathode deposit. He had known a case of a hydrogen vacuum tube made of lead glass and provided with outside electrodes that was used as a detector for high-frequency oscillations, which failed to work after long continuous running, owing to the disappearance of the hydrogen, but worked again if the tube was left of one side for a week.

Mr. C. E. S. **Phillips** thought the electrodes played an important part in the diminution of pressure on working. Mr. Hill's results seem to indicate that the glass may become aged. It might be possible to age the glass for X-ray bulbs artificially. The constant change in hardness of X-rays owing to the change of pressure was a most serious drawback especially in medical work, where it was holding back the progress of the applications of X-rays to medicine.

Dr. G. W. C. **Kaye** remarked that in X-ray bulbs the spluttering of the cathode absorbed the gas. Campbell Swinton found some years ago that by heating the walls of a vacuum tube small bubbles of gas were given off, which was mostly hydrogen, and found that they came from distances up to 0·15 mm. below the surface. Ramsay and Collie had recently found helium to be evolved in a similar case. It was difficult to explain these results on the penetration theory, as the charged atoms would not penetrate the thinnest aluminium foil. This theory also does not explain the differences in behaviour of different kinds of glass. The results may be due to chemical activity excited in the gas by the discharge, such as has recently been found by Prof. Strutt to be the case with nitrogen. The violet coloration often noted in the glass of vacuum tubes was due to a sub-oxide of sodium. He suggested Mr. Hill experimented with silica bulbs.

Mr. A. A. **Campbell Swinton** contributed the following remarks : Mr. Hill alludes to my Paper published in the "Proceedings" of the Royal Society, Series A, Vol. LXXIX., 1907, but does not seem to be aware of my further Paper in the "Proceedings" of the Royal Society, Series A, Vol. LXXXI., 1908, in which the criticisms of Soddy and Mackenzie and of others are dealt with. In Mr. Soddy's experiments, as also in those of Mr. Hill, forms of electric discharge were employed with which the amount of heat communicated to the glass would be very great as compared with the amount of cathode-ray bombardment; whereas, in the arrangements adopted in my experiments, the converse was the case. Consequently, I do not think that the cause of the absorption of gas is necessarily at all the same in the one case as in the other, particularly as in my crucial experiment I used helium, which does not combine chemically with anything at ordinary temperatures.

The **Author** remarked that he would be glad to carry out the experiments suggested.

IV. *On a Method of Measuring the Thomson Effect.* By H. Redmayne Nettleton, *B.Sc., Assistant Lecturer in Physics at Birkbeck College.*

Received October 8, 1912. Read November 8, 1912.

1. Introduction.
2. Theory of Method.
3. Description of Apparatus.
4. Practice of Method.
5. Results of Experiment.
6. Discussion of Method and Results.

1. *Introduction.*

(*a*) When a current of electricity passes down a conductor in which a temperature gradient is maintained, the stationary value of the temperature at any place is dependent to a small extent on the direction of the current, for heat is evolved or absorbed in accordance with the Thomson effect.

Lord Kelvin expressed the relation between the heat evolved or absorbed, the temperature gradient and the current in the form $dQ=C\sigma \cdot d\theta$, where dQ is the heat evolved or absorbed per second by a current, C, in passing between two sections differing in temperature by $d\theta$. The quantity σ which may thus be measured in calories per 1°C. per ampere-second is called the specific heat of electricity.

Relative values of the Thomson effect in different metals have been obtained by Le Roux,[*] Trowbridge and Penrose,[†] and Battelli,[‡] the last named experimenter indeed attempting to obtain an absolute value by measuring directly the heat produced by the effect in a given time in a definite portion of a bar under temperature gradient. It must be remembered, however, that the heat produced by the effect at once spreads down the conductor and influences the quantity of heat propagated by conductivity, time being required for a steady state to be obtained, and it can be shown that, besides the intrinsic difficulty of measuring directly such a small quantity of heat as that produced by the effect, Battelli's method does not fall into harmony with the differential equation of distribution of temperature down a bar conveying an electric current—an equation first obtained by Verdet and given in his "Theorie Méchanique de la Chaleur." Haga[*] has devised a method, very general in its application, by which the specific heat of electricity can be obtained in absolute measure by comparing the change of temperature at a point when a current flowing down a conductor under temperature gradient is reversed with the rise in temperature produced by a current when the conductor is at uniform temperature throughout. Absolute measurements of the Thomson effect based on this method or on modifications of it have been made by Laws,[†] Lecher,[‡] Schoute,[§] Berg,[‖] and Aalderink,[¶] while Callendar has devised a method based conjointly on Verdet's equation and the variation of electrical resistance with temperature which has been successfully carried out by King.[**]

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[*] Le Roux, "Ann. de Chimie et de Phys.," X., p. 258 (1867).
[†] Trowbridge and Penrose, "Phil. Mag.," 3 Ser., Vol. XIV., p. 440, 1882.
[‡] Battelli, "Accad. delle Sci. di Torino, Atti," Vol. XXII., p. 548, 1887.

The object of the present communication is to show: (*a*) That the modification of temperature gradient due to the heat produced or absorbed by the Thomson effect in a conductor of uniform cross-section passing through two constant temperature sources is exactly similar to the modification of gradient which would be produced by a definite slow uniform movement of the conductor itself in the proper direction.

(*b*) That for exact similarity of disturbance of gradient by impressed velocity and Thomson effect respectively—or for exact neutralisation—the following simple relationship holds:

$$\frac{\text{Current of electricity in amperes}}{\text{Material flow of conductor in grammes per second.}} = \frac{\text{Specific heat of electricity.}}{\text{Specific heat of material of conductor.}}$$

(*c*) That by working in the following manner the impressed velocity method may be applied to mercury to measure the absolute value of the specific heat of electricity, however large the Joule effect may be or whatever the emissivity loss:—

1. A current of C amperes is passed down a column of mercury heated at the top and maintained cold at the bottom. A

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[*] Haga, "Ann. de l'École Polyt. de Delft," I., p. 145, 1885; III., p. 43 1886.

[†] Laws, "Phil. Mag.," Vol. VII., p. 560, 1904.

[‡] Lecher, "Ann. d. Physik," XIX., p. 853-867, 1906.

[§] Schoute, "Archives Néerlandaises," Série II., p. 175, 1907.

[‖] Berg, "Ann. d. Physik," XXXII.-XXXIII., pp. 477-519, June, 1910.

[¶] Aalderink, "Archives Néerlandaises," Série II., p. 321, 1910.

[**] King, Amer. Acad., Proc., XXXIII., p. 353, 1893.

thermo-junction at a point near the middle of the gradient registers the temperature. On reversing the current the temperature is slightly raised, say, by $\Delta\theta_1$.

2. The current being still maintained (reversed) to keep the Joule effect and emissivity constant, a flow of mercury of, say, *m* grams per second is started up the tube, the temperature falling when the steady state is attained by $\Delta\theta_2$.

It can then be shown that :

$$\frac{2C\sigma}{ms}=\frac{\Delta\theta_1}{\Delta\theta_2},$$

and so σ is easily calculated.

The ratio of the two temperature changes can easily be made the ratio of two galvanometer deflections. By the use of two vertical columns in the form of an inverted U the sensitiveness is increased and the thermo-electric work is much simplified.

2. *Theory of the Method.*

Consider a vertical column of mercury flowing down a uniform glass-tube, heated at the top and cold at the bottom. The flow is very slow, so that the temperature changes produced by it are very small, and the character of the isothermals is not appreciably altered by it, in which case it is immaterial whether the flow is parabolic or otherwise.* A current C amperes is also passing down the column.

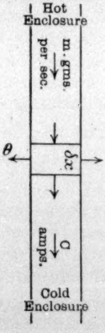

FIG. 1.

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*Nettleton, "Phil. Mag.," p. 590, April, 1910.

Let K=thermal conductivity of mercury.
s=specific heat of mercury.
σ=specific heat of electricity in mercury.
A=cross-section of the tube.

Let m=mass of mercury crossing a section of the tube per second.

The heat which enters per second a small prism fixed in space of depth δx and temperature θ is :—

(*a*) Due to conductivity $-KA\frac{d\theta}{dx}$.

(*b*) Joule effect heat. If S=specific resistance of mercury, and α be the temperature coefficient of resistance, the resistance of the prism $=S(1+\alpha\theta)\delta x/A$; so if J is the number of joules equal to a calorie the heat produced in this way

$$=\frac{C^2S}{J}(1+\alpha\theta)\frac{\delta x}{A}.$$

(*c*) Thomson effect heat $=-C\sigma\frac{d\theta}{dx}\delta x$

(σ is assumed to be positive).

(*d*) Flow effect heat $=ms\theta$.

The heat leaving the prism per second is :—

(*a*)′ Due to conductivity $-KA\frac{d}{dx}\left(\theta+\delta x\frac{d\theta}{dx}\right)$.

(*b*)′ Flow effect heat $=ms\left(\theta+\delta x\,.\,\frac{d\theta}{dx}\right)$.

(*c*)′ Due to emissivity $=Ep(\theta-\theta_0)\delta x$,

where θ_0=temperature of the enclosure,
p=perimeter of tube,
E=Newtonian coefficient of emissivity.

Equating the heat entering to the heat leaving, we obtain the differential equation pertaining to the steady state, which may conveniently be written—

$$\frac{d^2\theta}{dx^2}+a\frac{d\theta}{dx}+b\theta=c,$$

where

$$a=-\frac{(C\sigma+ms)}{KA},$$

$$b=\frac{1}{KA}\left[\frac{C^2S\alpha}{JA}-Ep\right],$$

$$c=-\frac{1}{KA}\left[\frac{C^2S}{JA}+Ep\theta_0\right].$$

If all temperatures be reckoned from the cold source as zero, θ_1 be the temperature of the hot source, and L the distance between the sources that is the entire length of the temperature-gradient, the temperature θ at a distance x from the higher source is given by the following general solution :—

$$\theta=\frac{c}{b}\left[\frac{e^{-\frac{ax}{2}}\sinh\lambda(x-\mathrm{L})-e^{-\frac{a(x-\mathrm{L})}{2}}\sinh\lambda x}{\sinh\lambda\mathrm{L}}\right]$$

$$-\frac{\theta_1 e^{-\frac{ax}{2}}\sinh\lambda(x-\mathrm{L})}{\sinh\lambda\mathrm{L}}+\frac{c}{b},$$

where $$\lambda=\frac{\sqrt{a^2-4b}}{2}.$$

If now we assume that the Thomson effect and flow effect are so small that the terms $\frac{ax}{2}, \frac{a\mathrm{L}}{2}$ are small compared with unity—and this condition is adhered to throughout the experiment—we may without making any approximation as to the magnitude of the "b" and "c" terms of the differential equation write the solution in the form,

$$\theta=a\mathrm{F}(\theta_1, x, \mathrm{L}, b, c)+f(\theta_1, x, \mathrm{L}, b, c),$$

or more shortly

$$\theta=a\mathrm{F}+f,$$

where F and f are functions of θ_1, x, L, b, c, but not of the "a" term in the differential equation.

This solution forms the basis of the method the values of F and f not being required as long as they are kept constant throughout. In general F and f are complex, but if it be assumed in addition that the emissivity loss is small and that consequently the term $\frac{b\mathrm{L}^2}{6}$ is small compared with unity a most useful solution may be obtained·

$$\theta=\theta_1\left(\frac{ax}{2}-1\right)\frac{x-\mathrm{L}}{\mathrm{L}}+\theta_1\left(\frac{ax}{2}-1\right)\left(\frac{x-\mathrm{L}}{\mathrm{L}}\right)\frac{bx(2\mathrm{L}-x)}{6}$$

$$+\frac{a}{2}(\mathrm{L}-2x)\tfrac{1}{6}cx(x-\mathrm{L})+\tfrac{1}{2}cx(x-\mathrm{L}).$$

This solution is most useful in showing the effect of the "b" and "c" terms, and in estimating the rise of temperature at a point due to reversing a current or due to superimposing a flow.

Case of Two Parallel Tubes.

Consider two parallel tubes, A and B, through which thermo-junctions are inserted near the middle of the respective gradients.

The temperature at " a " when a current is passing down the tube is given by $\theta = aF + f$, and at " B " likewise when a current or flow is passing down by $\theta' = a'F' + f'$. When a current is passing down A it will be passing up B, and hence if μ is the ratio of the cross-section of A to the cross-section of B, for all values of current and flow we have

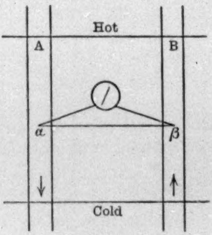

FIG. 2.

$$a' = -\mu a.$$

We may now easily express the difference of temperature between " a " and " β " for the following three cases, making use of the suffixes c, v to indicate respectively an electric current or flow of mercury down A and hence up B.

State of Electric Current or Flow.	Temperature of α.	Temperature of β.	Difference of Temperature between α and β.
(1) Current down A and up to B	$\theta_c = a_c F + f$	$\theta'_{-c} = -\mu a_c F' + f'$	$\Delta\theta_c = a_c(F + \mu F') + f - f'$
(2) Current up A and down B	$\theta_{-c} = -a_c F + f$	$\theta'_c = \mu a_c F' + f'$	$\Delta\theta_{-c} = -a_c(F + \mu F') + f - f'$
(3) Current up A and down B; Flow of mercury down A and up B	$\theta_{v-c} = (a_v - a_c)F + f$	$\theta'_{-v+c} = \mu(-a_v + a_c)F' + f'$	$\Delta\theta_v^{-c} = (a_v - a_c)(F + \mu F') + f - f'$

Now, $\Delta\theta_c - \Delta\theta_{-c}$ is measured directly by the deflection of the galvanometer spot on the scale ; call this deflection d ; it is the deflection due to reversing the current.

Similarly, $\Delta\theta_v^{-c} - \Delta\theta_{-c}$ is the deflection caused by superimposing a flow on the reversed current ; call this deflection d_v.

Then $$d_c = 2a_c(\mathrm{F} + \mu\mathrm{F}')$$

and $$d_v = a_v(\mathrm{F} + \mu\mathrm{F}'),$$

and hence $$\frac{d_c}{d_v} = \frac{2a_c}{a_v} = \frac{2\mathrm{C}\sigma}{ms}$$

or $$\sigma = \frac{s}{2\mathrm{C}} \times \frac{m}{d_v} \times d_c.$$

In each experiment several values of d_c and of m/d_v are found, and the mean of each set used in the calculation for σ.

3. *Description of Apparatus.*

The apparatus, which has been greatly modified during the progress of the research, will be seen on reference to Fig. 3 to consist essentially of two parallel glass experimental tubes T_1, T_2, clamped at their lower ends within the large zinc vessel V, and passing above into an annular heater, A, near the top of which they are joined by an inverted U piece. The left-hand tube T_1 is joined below by wide pressure tubing to the glass head H, while the right-hand tube T_2 is similarly connected to the glass bulb tube B, and thence by narrow pressure tubing to the system of capillaries, which form the flow resistance. The mercury, which fills the apparatus, may be made to convey an electric current, which enters by the head H, and leaves the bulb tube B by eight platinum wires fused through the glass and dipping below into the mercury reservoir D. By releasing the pinch cock F, the mercury may be caused to flow slowly from the head H, through the apparatus and the attached capillaries. When the vessel V is full of cold water, and steam is circulating through the annular space of the heater A, a temperature gradient, about 8 cms. long, is obtained in the middle of the experimental tubes T_1 and T_2, temperature changes near the middle of each gradient being measured by the thermo-junctions T_{J1}, T_{J2}, of iron and constantan sealed through the glass tubes.

The following is the method of preparing and mounting the experimental tubes, great care being taken in the insertion and insulation of the thermo-junctions. A cane of glass, as uniform in cross-section as possible, the internal diameter being just under 7 mm. and the external diameter about 1 cm., was divided into four parts. Two of the parts were prepared, as shown in Fig. 4, by Messrs. A. C. Cossor, Ltd., the iron

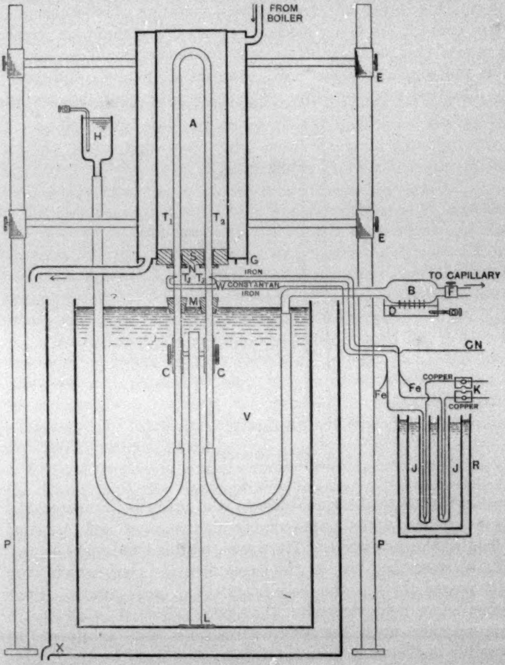
FROM BOILER
A
H
E
E
T_1
T_2
G
S
N
IRON
T_1 T_2
W
CONSTANTAN
IRON
M
TO CAPILLARY
B
F
D
CN
COPPER
Fe
K
COPPER
Fe
C
C
V
J
J
R
P
P
L
X

FIG. 3.

and constantan wire being passed through glass beads, the junction formed, and the insertion effected with a minimum upset of internal cross-section. The wires were in the same vertical plane, the junction being about 3·5 cms. from the nearer end of the tube and some 25 cms. from the further end.

Several such tubes have at different periods been supplied to the author, and the glass work brought to such perfection that in the tubes used in the present apparatus internal distortion was hardly visible. The next procedure was the insulation of the wires within the tubes, which was effected first with a thin layer of black club enamel, which flows very easily, and then, after drying with hot air, with a coating of red household hard enamel, use being made of a very fine camel-hair brush. Practice was required before neat insulation could be carried out and rough or unsuccessful coatings of enamel could easily be dissolved away by ether. The tubes, after thorough drying, were returned to the glass-blower, who skilfully joined on the top portions with inappreciable disturbance and without affecting the insulation. The pressure tubing and corks M_1, n_2, M_2, n_2 were then put on the experimental tubes, which were also passed through the loose holes of the large cork S. Both tubes being held vertically, they were adjusted relatively to one another until the thermo-junctions were as nearly as possible in the same horizontal line : thus fixed by the double clamp C they were returned to the glassblower to be joined together by a U piece.

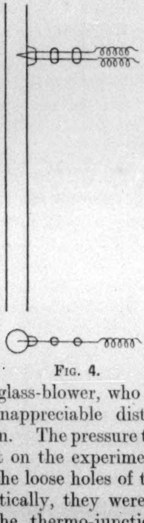

FIG. 4.

The apparatus was now filled with mercury, which had been distilled and purified by falling in fine drops through dilute nitric acid, and the clamp holding the tubes was then itself clamped at a convenient height on the firm iron upright fixed within the zinc vessel V. The annular heater was now lowered down its supporting pillars P_1, P_2, towards the cork S, which could easily be made to fit it by adjusting the position of the outer vessel X and its contents. The heater and cork "S" were then lowered to the final position, the corks n_1, n_2 being brought up against the cork S, and closing the space above to air currents. Cotton wool was carefully wound round the tubes between the corks as well as round the annular heater, underneath which were soldered concentric strips of gauze forming a hold for cotton wool, which could be packed into the spaces lying between them and the outside of the cork S. The vessel V was filled with cold water up to the middle of the corks M_1, M_2, which had previously been boiled in paraffin wax. By covering the water with a layer of Fleuss-pump oil its evaporation into the cotton wool above was prevented.

The two constantan wires were joined together at W, and the two iron wires lead away to the copper leads shown in the diagram, the junctions of copper and iron being side by side in two glass tubes J, J, immersed in water in the Newton annular cooling calorimeter R. The ends C*n*, Fe_1, Fe_2 of the wires were of use only when the actual temperature of the junction T_{J1}, T_{J2} were required. The iron constantan couple has not only the merit of being suitable for fusing through glass, but also possesses a remarkably high thermo-electric power—some 50 microvolts per degree centigrade—the relation between electromotive force and temperature being for the range 0°—100°C. strictly linear.

The galvanometer was a quartz fibre mirror instrument of the Broca type, the equivalent resistance of the two coils in parallel being 8·28 ohms, and the sensitiveness such that a micro-ampere produced about 11 cms. deflection on the scale distant nearly 2 metres from the mirror. The resistance of the rest of the circuit was 1·18 ohms, and a degree alteration in difference of temperature between the thermo-junctions was represented by 60 cms. deflection. There was no movement of the galvanometer mirror on quickly reversing a current of 8 amperes whether the galvanometer circuit was open, closed, or short-circuited, thus showing both that the instrument was sufficiently far away from the experimental tubes to render magnetic disturbance negligible, and that the insulation of the thermo-junctions was satisfactory.

An operation of importance is the adjustment, when steam is passing through the heater of the thermo-junctions T_{J1}, T_{J2}, to the same temperature, so that when the galvanometer circuit is closed the line of light is suitably situated on the scale. It was anticipated that this would be troublesome, and earlier apparatus was so designed that each experimental tube could be raised or lowered relatively to the other Experience showed, however, that this was absolutely unnecessary, for if the junctions were set by eye to be as nearly as possible in the same horizontal line, the fine adjustment could easily be made by reducing the emissivity loss on one tube or the other by the addition of a small piece or two of cotton wool, inserted with the aid of a knife or prong, to that already round the tubes between the corks. This method of adjustment is not only convenient, but also brings out most clearly the need for complete firmness and rest of the wool during an experiment, emphasising the necessity for protection from draughts, convection eddies and vibration.

The currents used during the research varied from 4 to 9 amperes, and were derived from the 100-volt mains in series with which were a woven-wire resistance with six steps, a 0–20 ohm rheostat in 40 steps, a standard $\frac{1}{100}$ ohm resistance, a hot-wire ammeter, and the reversing commutator and main apparatus. In multiple arc with the woven-wire resistance were a water resistance and 0–1 Weston ammeter, the water resistance in which the distance between the zinc electrodes could be varied forming a most suitable fine adjustment for the current in the main circuit. During the night the current from the mains was much steadier than during the day, but prior to any reading the current was for some time closely watched and regulated if necessary by the water rheostat.

It remains to describe the method of obtaining a constant flow, which will cause temperature changes of the same order as those produced by reversing a current of 5 or 6 amperes. Calculation shows that a flow of 1 gram per hour is about the equivalent of reversing 3 amperes, while before it was decided to proceed with the method, preliminary experiments made two years ago showed that flows of mercury of about 2 grams per hour could be obtained constant to 1 per cent. without any special precautions. Most of the flows used in the apparatus described were as large as 3·5 grams per hour and, at least, constant to one part in 150, but, what is more important, the ratios mean flow to galvanometer deflection produced by the flow were at least constant to 2 per cent., which is as high an accuracy as can be obtained in the value of the galvanometer deflections produced by current reversals. It is interesting to note that an outflow of 5 grams per hour, which is faster than any used in the research, would roughtly mean a linear velocity through the experimental tubes of about 1 cm. per hour. The flows then being very small, it is unnecessary to maintain a constant head by any overflow arrangement, but the head tube H was allowed a width of about 5 cms. The resistance consisted of about 2 metres of thick-walled thermometer tubing of fine bore, on to which was joined by pressure tubing a piece of ordinary glass tubing drawn out into a very fine capillary over a metre long. The last and finest portion of this capillary lay on a wooden inclined plane, by altering the inclination of which the value of the flow could be modified. The drop period varied from about 4 to 8 seconds, according to the magnitude of the flow which was measured, as described in the next section.

4. *Practice of the Method.*

An experiment was usually started between three and four o'clock in the afternoon by preparing the boiler and starting the flow of steam through the annular heater. The current was started after steam had been passing for an hour, and after another hour the short-circuit plug of the galvanometer key was removed and the position of the line of light on the galvanometer scale examined and modified, if need be, by the addition of a little piece of cotton wool to one side or other of the surroundings of the experimental tubes. Even after the attainment of the steady state the galvanometer needle still showed signs of irregular movement, but towards 10 at night the deflection became much steadier, and it was possible to proceed with the experiment. This "zero movement" has caused the author, and many other experimenters on the Thomson effect, the greatest trouble; in earlier apparatus used in the research it was large and progressive, and was attributed in part to the warping of wood which supported the annular heater and glass tubes. In the present apparatus the supports were all very stable; convection currents from the heater were allayed by the free use of cotton wool, and humidity changes within the wool were prevented by covering the water in the cooling tank with a layer of Fleuss pump oil. So great was the improvement brought about by the modification of the apparatus and by working at night that often, though not always, the needle would behave almost perfectly, good behaviour being illustrated by the reversal readings of Experiment IV., given in the next section, and relatively bad behaviour by the reversals of Experiment VI.

The deflection having become steady, the statical part of the experiment or " the reversals " was proceeded with as follows: The needle of the ammeter in the main circuit was maintained by the water resistance in one of the branch circuits accurately on the current reading chosen, which was always an exact number of amperes, for some 10 or 15 minutes prior to taking the galvanometer deflection. When this had been read the commutator was reversed and the new steady state waited for—a period of about 35 minutes, during the last 10 of which the movement was very little. A uniform period of 40 minutes was allowed between each reversal, the current being closely regulated for 10 minutes before each reading. The deflections due to reversing the current were then found by subtracting the mean of two consecutive readings with the current in the same direction from the intermediate reading with the current in the opposite direction.

Sufficient reversal readings having been obtained, and the current being in the direction causing the line of light to lie on the right-hand side of the scale, the dynamical or flow part of the experiment was proceeded with by observing the deflection as before, short-circuiting the galvanometer, and releasing the pinch-cock whereby a flow of mercury through the apparatus could be started. The release of the compressed pressure-tubing evidently caused a momentary outrush of mercury, for, unless the galvanometer was short-circuited, the line of light was thrown off the scale towards the left. For this same reason, too, the velocity of the mercury was not measured until the flow had settled down.

The method of obtaining the flow readings and flow magnitude is best illustrated by an example :—In Experiment I. the last reversal reading was at 1·0 a.m., when the deflection was 9·75 to the right. The galvanometer was at once short-circuited and the pinch-cock released very small drops of mercury commencing to fall from the end of the fine capillary. By 1·15 it was safe to re-open the galvanometer circuit, the line of light being on the scale, but well to the left. At 1·25 the measurement of flow was commenced, a weighed watch-glass being placed under the end of the capillary as soon as the first drop after 1·25 had fallen. Meanwhile the deflection had approached a steady value, which, after 10 minutes watching of the current, was read immediately after 1·42 as 19·7 to the left; the galvanometer was then short-circuited. The watch-glass was then carefully removed the moment the first drop after 1·45 had fallen, and the pinch-cock was then at once screwed up tightly. After 2 o'clock the galvanometer circuit was opened, and the final reading taken at 2·25, after steadying the current, was 10·0 to the right. The intervals between galvanometer readings were thus practically equal, and the mean of 9·75 and 10·0 added to 19·7 gave the flow deflection 29·58 corrected for zero change. This was now divided into 1·0674, the number of grammes of mercury outflowing in the 20 minutes, the values of M/*d* in the tables below thus being obtained. The above process was repeated, usually three times, using flows of different value, the modification of flow being easily effected by raising or lowering the inclined plane on which the end portion of the fine capillary rested.

That different values of M/*d* should be obtained on different days is only to be expected from the theory which shows that it will depend slightly on the current and emissivity alterations, but more largely and directly on the difference of temperature between the hot and cold sources. Similarly, in the statical experiment the ratio C/*d* is only theoretically and practically an approximate constant. Any change, however, which affects the ratio M/*d* will equally affect the ratio C/*d*, so that changes of temperature in the room would in no way impair the accuracy of the values obtained, provided the second part of the experiment follows on immediately after the first.

At the conclusion of the flow portion of the experiment the actual temperature of each thermo-junction was found by balancing its electromotive force against that of a similar junction tied to a good standardised thermometer immersed in hot water in a long cylindrical vacuum vessel.

The actual current values taken were not based on the readings of the hot-wire ammeter but on the electromotive force across the standard $\frac{1}{100}$ ohm resistance as measured on a Paul millivoltmeter, which could not always be spared from the laboratory until late in the evening. This millivoltmeter was most carefully calibrated, both before and after the series of six experiemnts, on a Crompton potentiometer against a new Weston cell, which agreed to one part in 400 with an older Clark cell.

The only other value required before calculating the value of σ is s the specific heat of mercury. This is taken as 0·03294, which is Barnes* value at 62°C. and which agrees very closely with the value given by Winkelmann.

5. *Results of Experiments*

The first table given is a verification of the equation $\theta = aF + f$ for flows, and shows that they are small enough to permit of the one approximation made in the theory of the method.

Hour.	Galvanometer readings.		Mean deflection d.	Mass outflowing in 20 min. M.	M/d.
9·55		19·55			
10·37	22·2		40·58	1·4885	0·0367
11·20		17·2			
12·2	21·5		38·70	1·4194	0·0367
12·45		17·2			
1·27	15·0		32·43	1·2024	0·0371
2·10		17·65			
2·52	6·3		24·15	0·8884	0·0368
3·35		18·05			
4·17	6·45		24·78	0·8984	0·0363
5·0		18·60			
5·42	2·65		21·75	0·7978	0·0367
6·25		19·60			
7·7	9·45		28·88	1·0682	0·0370
7·50		19·25			

In this experiment, which proves that the flow of mercury is proportional to the temperature change it produces, no electric current was running.

The results of six consecutive experiments using differing currents are now given :—

Experiment I.

Date June 12–13, 1912.
Temperature of thermo-junctions, 60·2°C.
Magnitude of current, 6·24 amperes.

TABLE A.—*Reversals.*

Hour.	Galvanometer readings.		Deflection d_c due to reversal of current.
7·0	9·3		
7·40		10·25	19·20
8·20	8·6		18·93
9·0		10·4	19·30
9·40	9·2		19·35
10·20		9·9	19·08
11·0	9·15		19·10
11·40		10·0	19·15
12·20	9·15		19·03
1·0		9·75	

Mean value of $d_c = 19·14$.

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* Barnes, Brit. Assoc. Report, p. 530, 1902.

Table B.—*Flows.*

Hour.	Galvanometer readings.		Deflection.	Mass M outflowing in 20 mins.	M/*d*.
1·0		9·75			
1·42	19·7		29·58	1·0674	0·0361
2·25		10·0			
3·7	9·4		20·20	0·7248	0·0359
3·50		11·6			
4·32	21·4		32·8	1·1552	0·0352
5·15		11·2			
5·57	19·85		30·95	1·1080	0·0358
6·40		11·0			

Mean value of M/*d*=0·0357[5.

whence
$$\sigma=\frac{0\cdot03294}{2}\times\frac{19\cdot14}{6\cdot24}\times\frac{0\cdot03575}{1200}.$$
$$=0\cdot00000151.$$

Experiment II.

Date June 14th and 15th.
Temperature of thermo-junctions, 61·4°C.
Magnitude of current, 7·24 amperes.

Table A.—*Reversals.*

Hour.	Galvanometer readings.		Deflection d_c due to reversal of current.
11·15		16·1	
11·55	6·75		22·70
12·35		15·8	22·43
1·15	6·50		22·40
1·55		16·0	22·50
2·35	6·50		22·68
3·15		16·35	

Mean value of d_c=22·54.

Table B.—*Flows.*

Hour.	Galvanometer readings.		Deflection.	Mass M outflowing in 20 mins.	M/*d*.
	Height of capillary tube altered by accident.		Not taken.	Not weighed.	...
4·40		17·45			
5·22	13·0		30·58	1·0707	0·0350
6·5		17·70			
6·47	13·25		31·18	1·1005	0·0353
7·30		18·15			
8·12	12·10		30·25	1·0810	0·0357
8·55		18·15			
9·37	12·85		30·93	1·0855	0·0351
10·20		18·0			

Mean value M/*d*=0·0353.

whence $\sigma=0\cdot00000151$.

Experiment III.

Date June 18th and 19th.
Temperature of thermo-junctions, 63·7°C.
Magnitude of current, 8·25 amperes.

TABLE A.—*Reversals.*

Hour.	Galvanometer readings.		Deflection d_c due to reversal of current.
10·10	10·1		
10·50		16·2	26·13
11·30	9·75		25·85
12·10		16·0	25·75
12·50	9·75		25·80
1·30		16·1	26·00
2·10	10·05		26·10
2·50		16·0	

Mean value of $d_c = 25·94$.

TABLE B.—*Flows.*

Hour.	Galvanometer readings.		Deflection.	Mass M outflowing in 20 mins.	M/*d*.
4·15		17·5			
4·47	19·15		36·6	1·3044	0·0356
5·40		17·4			
6·22	12·85		30·6	1·0908	0·0356
7·5		18·1			
7·47	13·9		31·8	1·1392	0·0358
8·30		17·7			
9·12	14·4		32·4	1·1703	0·0361
9·55		18·3			

Mean value of $M/d = 0·0358$.

whence $\sigma = 0·00000155$.

Experiment IV.

Date June 21st and 22nd.
Temperature of thermo-junctions, 61·3°C.
Current, 5·12 amperes.

TABLE A.—*Reversals.*

Hour.	Galvanometer readings.		Deflection d_c due to reversal of current.
10·0		16·65	
10·40	1·05		15·60
11·20		16·65	15·55
12·0	0·95		15·60
12·40		16·45	15·50
1·20	0·95		15·60
2·0		16·65	

Mean value of $d_c = 15·57$.

TABLE B.—*Flows.*

Hour.	Galvanometer readings.		Deflection.	Mass M outflowing in 20 mins.	M/d.
2·0		16·65			
2·42	5·3		22·25	0·8091	0·0364
3·25		17·25			
4·7	6·9		23·53	0·8586	0·0365
4·50		17·8			
5·32	13·95		31·9	1·1653	0·0365
6·15		18·1			
6·57	14·65		32·68	1·2055	0·0369
7·40		17·95			

Mean value of $M/d=0.0366$.

whence $\sigma=0{\cdot}00000153$.

Experiment V.

Date, June 26th and 27th.
Temperature of thermo-junctions, 60·6.
Magnitude of current, 4·10 amperes.

TABLE A.—*Reversals.*

Hour.	Galvanometer readings.		Deflection d_c due to reversal of current.
11·10		17·85	
11·50	5·45		12·28
12·30		17·6	12·22
1·10	5·30		12·30
1·50		17·6	12·20
2·30	5·5		12·33
3·10		18·05	

Mean value of $d_c=12{\cdot}26[6$.

TABLE B.—*Flows.*

Hour.	Galvanometer readings.		Deflection.	Mass M outflowing in 20 mins.	M/l.
3·10		18·05			
3·52	9·2		27·98	1·0351	0·0370
4·35		19·5			
5·17	21·3		40·44	1·5128	0·0374
6·0		18·78			
6·45	18·2		37·14	1·3953	0·0376
7·25		19·1			
8·8	19·65		33·45	1·4435	0·0375
8·50		18·5			

Mean value of $M/d=0.0374$.

whence $\sigma=0{\cdot}00000154$.

Experiment VI.

Date, June 28th and 29th.
Temperature of thermo-junction, 67·5°C.
Magnitude of current, 9·26 amperes.

TABLE A.—*Reversals.*

Hour.	Galvanometer readings.		Deflection d_c due to reversal of current.
9·10	9·4		
9·50		19·0	28·95
10·30	10·5		29·32
11·10		18·63	29·16
11·50	10·55		29·44
12·30		19·15	29·38
1·10	9·90		29·48
1·50		20·0	29·33
2·30	8·75		28·95
3·10		20·4	

Mean value of d_c=29·25.

The zero movement was worse than usual at night.

TABLE B.—*Flows.*

Hour.	Galvanometer readings.		Deflection.	Mass M outflowing in 20 mins.	M/*d*.
3·10		20·4			
3·52	13·45		34·85	1·2815	0·0367[7
4·35		22·4			
5·17	16·95		39·29	1·4426	0·0367[2
6·0		22·28			
6·42	18·58		41·32	1·5220	0·0368[3
7·25		23·2			
8·7	16·0		38·98	1·4295	0·0366[7
8·50		22·75			

Mean value of M/*d*=0.03675.

The flows used were all large and more constant than usual. This experiment gives σ=0·00000159.

The results are summarised below :—

Current in amperes.	Temperature.	Value of σ is calories per degree Centigrade per coulomb.
4·1	60·6	0·00000154
5·12	61·3	0·00000153
6·24	60·2	0·00000151
7·24	61·4	0·00000151
8·25	63·7	0·00000155
9·26	67·5	0·00000159

On ascertaining the direction of the current the Thomson effect was seen to be negative for a flow of mercury down one of the experimental tubes from hot to cold had a similar effect to reversing a current initially passing downwards.

6. *Discussion of Method and Results.*

While the theory of the method is general, it would seem limited in practice to mercury and amalgams. Again it is more important than in Haga's method that adequate time be allowed for the attainment of the steady state; for if the time were insufficient the deflections due to current reversal would be too small, while the deflections due to impressed velocity would be larger than their ultimate value and so too low a value of the effect would be obtained. Probably this objection would disappear if the flows were started by slowly opening a tap.

On the other hand, the flow effect and Thomson effect are similar terms in the equation of the method, their influence on the emissivity loss being similar and negligible; they can, moreover, be made to give deflections of the same order; on the contrary, the Thomson effect and Joule effect are dissimilar and dependent, and in Haga's method in which a comparison between these two effects is made a measurement of temperature gradient is necessary, and deflections of the same order can only be obtained at the expense of differences in emissivity conditions. In practice the flow portion of the experiment has never presented the slightest difficulty, but much trouble has been taken in attaining suitable deflections on reversing the current and, above all, in securing consistency in their value by reducing to a minimum variations of zero.

The author believes that the method will be very suitable for finding the variation of Thomson effect with temperature, but has not attempted it with the present apparatus in which only the higher temperature source could conveniently be raised. Schoute's results indicate that the variation with temperature is large, and it is thus not advisable to determine the temperature coefficient by increasing the temperature-gap. Moreover, it is desirable to shorten the time needed for attainment of the steady state.

The results obtained by the author are in fair agreement with those obtained by Schoute, employing a modification of Haga's method. Schoute obtained values, somewhat inconsistent, varying from as low as 0·00000134 at 32°C. to as high as 0·00000248 at 154°C. the value changing from 0·00000160 at 53°C. to 0·00000180 at 58°C. The only other experimenter on the effect in mercury is Haga, who, working as far back as 1885, obtained a value as low as 0·00000069 at 78°C., but owing to the difficulties under which he worked his results are very inconsistent; moreover, the Joule effect and Thomson effect temperature changes were measured under different conditions and in different ways, and what is serious in this particular method a very large upset of cross-section was caused by the introduction of his thermo-junctions.

In conclusion, the author must express his thanks to Principal Armitage-Smith, for facilities for carrying out the work, all of which was done through the night; to Dr. A. Griffiths, head of the Physics Department, for his great interest, and to Mr. J. L. Prescott, who, while at Birkbeck College, made valuable translations of most of the foreign Papers referred to in the introduction.

ABSTRACT.

In this Paper an investigation is made of the distribution of temperature down a conductor conveying an electric current and at the same time moving uniformly through two fixed temperature sources. The effect of the Thomson heat on the distribution is seen to be exactly similar to the effect of a small impressed velocity. This result is applied to mercury to measure the Thomson effect by comparing the alteration of temperature $\Delta\theta_1$ at a point near the middle of the gradient caused by reversing a current of C amperes with the alteration of temperature $\Delta\theta_2$ at the same point due to a flow of mercury of m grammes per second. It is shown that, without any approximation as to emissivity loss or magnitude of Joulian heat, $2C\sigma/ms = \Delta\theta_1/\Delta\theta_2$, where s is the specific heat of mercury and σ the specific heat of electricity. Working with currents of from 4 to 9 amperes and with flows of different magnitudes—but never exceeding 1 cm. per hour—consistent values of σ are obtained, the value at 61°C. being $-1{\cdot}52 \times 10^{-6}$ calories per degree Centigrade per coulomb. The thermo-junctions, which were of iron and constantan, were fused through the glass tubes with inappreciable distortion.

DISCUSSION.

The PRESIDENT stated that the Paper dealt with a difficult problem and gave an adequate and promising method of measuring the Thomson effect, but he queried whether it was justified to assume the velocity of flow was constant over the cross-section of the tube.

Dr. A. GRIFFITHS stated that it was not assumed that the velocity over the cross-section was constant, but only that the temperature was constant, which on account of the extreme slowness of the flow would be justified. The author performed one experiment when the flow was stopped and obtained the same difference in temperature on reversing the current.

Dr. W. E. **Sumpner** pointed out that it was not realised how extremely slow the flow was—something of the order of 1 cm. per hour.

Prof. C. H. **Lees** was struck with the ingenuity of the method. There were, however, a number of small corrections to be considered, such as the heat transmitted through the glass. More accurate knowledge of the thermoelectric phenomena in liquids was urgently needed.

Mr. R. S. **Whipple** inquired how the iron and constantan wires were fused into the glass tube to which the Author replied.

V. *An Improved Joule Radiometer and its Applications.* By F. W. Jordan, *A.R.C.S., B.Sc.*

Received October 21, 1912. Read November 8, 1912.

I have already described a simple Joule apparatus* adapted to show the existence of the Peltier and Thomson effects. The tube and partition in that apparatus possessed a considerable thermal capacity and were made of bad thermal conductors. The time of attaining temperature-equilibrium was consequently excessive, and its extreme sensibility to stray heat caused the zero to creep in such a way as to render the apparatus unsuitable for exact measurements. The following apparatus was designed to eliminate the necessity of elaborate thermal insulation and to give a steady zero and a fairly high sensibility.

A brass tube A (Fig. 1), 4 cm. long, 1·6 cm. diameter and 0·2 cm. thick, was divided longitudinally into two compartments by a copper plate B, 0·6 mm. thick. Two rectangular gaps were cut in diagonally opposite corners of the partition and two sector-shaped discs of copper, *cc*, were soldered to the edges of the gaps to form the horizontal sides of the channels for the current of air between the compartments. The partition B and the discs *cc* were soldered to the inner surface of the brass tube A. Two light mica vanes, *dd*, each 7 mm. by 6·5 mm. and total weight 1·4 mgm., were fixed with a little shellac to a fine glass stem *e*. The distance between the vanes and the other dimensions were arranged so that when suspended the clearance between the edges of the vanes and the sides of the channels was about 1 mm. A small silvered glass mirror *m*, 3·5 by 2·5 by 0·2 mm., was attached to the glass stem *e*, and the whole system was suspended by a quartz fibre 9 cm. long and 0·004 mm. diameter about. The flanged ends of the brass tube A and the brass cover plates DD were ground plane and screwed together tightly. The glass tube E carrying the torsion head was fitted over a short brass tube fixed centrally to the upper cover plate. The motion of the vanes was limited by stops to about 20 degrees. The air in the interior of the apparatus was thus practically sealed from communication with the atmosphere and shielded from stray heat by the brass enclosure. The apparatus was fitted inside a concentric brass tube and the whole was mounted on a levelling stand. More elaborate thermal insulation would probably be required in the neighbourhood of intense sources of heat and also for a more sensitive instrument.

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* "Nature," May 18, 1911, p. 380.

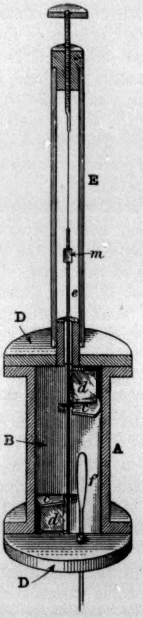

FIG. 1.

The sensibility of an instrument of this type may be expressed by the deflection in millimetres on a scale at a distance of a metre from the mirror produced by the absorption or evolution of heat at the rate of a microwatt in one of the compartments. This was determined by passing a measured current through a known resistance in one of the compartments.

The ends of a short length of No. 47 eureka wire *f* were soldered to two leads *g* of No. 36 copper wire. The insulated leads *g* were fastened together, and to the inside of a narrow glass tube with paraffin wax. The glass tube was passed through a circular aperture in the lower cover plate and sealed with chatterton compound ; a pair of dummy leads were similarly mounted in the other compartment to compensate any gain or loss of heat along the leads to the resistance loop.

The position of the spot of light on the scale was steady a few minutes after the apparatus had been levelled and adjusted. Bringing the hand quite near to the outer brass tube had no appreciable effect, and the zero remained steady to within a fraction of a millimetre. The motion of the vanes was excessively damped, and the period of swing was much greater than the undamped natural period.

The deflections produced by measured currents through the resistance loop were very nearly proportional to the squares of the currents. This agreed with the results I have obtained with other instruments of this type. In the case of the full scale deflection the spot of light became steady 40 seconds after the current was started through the resistance. On breaking the current the spot of light returned to the original zero in about the same interval of time. This interval of time could be reduced by diminishing the air resistance to the motion of the vane. The latter could be effected by increasing the clearance around the edges of the vanes. The following figures give some idea of the sensibility of the instrument.

A deflection of 122 mm. on a scale at a distance of a metre from the mirror was produced by a current of 4·75 milliamperes through the resistance of 10·5 ohms. This gives a sensibility of 0·52 mm. per microwatt. The small deflection of about 1 mm. produced by 0·5 milliampere could be repeatedly observed. The range of the instrument could be extended to measure feeble alternating currents of the order of 10^{-4} amperes by using a heater of 1,000 ohms, and the least detectable current in this case would be 5×10^{-5} amperes.

Alternating currents producing more than the full scale deflection could be conveniently and accurately measured by the compensation methods of H. L. Callendar* and K. Ångström.† In Callendar's method the heat from the heater would be absorbed by the current through a thermo-junction in the same compartment. In Ångström's method the heat would be balanced by the heat from a similar resistance in the other compartment. A calibration with direct currents would be necessary in every case to measure alternating currents.

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* Phys. Soc. "Proc." Vol. XXIII., March 17, 1911.

† Phys. "Zeit.," p. 685, 1905.

Although I have not yet tried it, the instrument may be adapted to measure the heat given out by small quantities of radium. A calculation shows that 1 mgm. of radium should give a deflection of 50 mm. on the scale.

To measure radiant heat it would be sufficient for most purposes to make a small window of rock salt or fluorite of about 50 sq. mm. in the side of the brass tube A and direct the radiant heat on to a thin metal disc supported centrally by a fibre in one of the compartments. The rate of absorption of heat by the disc could be measured by Callendar's method. The indications of the instrument could thus be made independent of radiant heat from directions other than those limited by a tube directed towards the window and the receiving disc.

Previous to this I had constructed on similar lines an apparatus in which the tube and partition were made of thin mica and 12 cm. long. This larger instrument was eleven times as sensitive as the one described here, but owing to the difficulty of thermally insulating the apparatus it was abandoned. There seems to be no reason why that sensibility should not be attained again in an improved form of instrument.

In the original Joule apparatus the radiation from an external source was absorbed by the badly conducting glass walls and partition of cardboard. The convection current of air was thus formed near the inner surface of the compartment and attained a steady velocity when temperature equilibrium had been established. The latter state would only be complete after a considerable interval of time, owing to the large thermal capacity and small thermal conductivity of the enclosure. In the modified apparatus the heat radiated to the walls of a compartment is practically ineffective in producing a convection current, and thus one cause of its slowness of action and unsteadiness of zero has been removed.

Since the construction of this experimental form of radiometer a compensation method of measuring the Thomson effect has occurred to me. The present apparatus is on too small a scale and would have to be considerably modified in order to be suitable for this measurement. The following is an outline of the method.

A thin wire of the metal AB (Fig. 2) is attached with solder or an electro-deposit of the same metal to the thick leads D and E, also of the same metal. These leads are insulated from, and led through, the lower cover plate of the radiometer. Two thin wires of different metals, OF and OG, are attached to wire AB to form a thermo-junction, O, for the measurement of the temperature at the centre of AB. One of these wires, OG, should be of the same metal as AB. Two small plates of mica, perforated to slip over AB, serve to separate the parts of the wire to the right and left of the thermo-junction and also to form a continuation of the partition of the radiometer.

Let an alternating current, C, or a direct current reversed at regular short intervals be passed through AB. The resistances of the two halves of the wire may be adjusted by electro-deposition, so that the convection currents on either side of the partition cause only a small deflection of the vane. This adjustment need not be made exactly, as balance can be effected by passing a current through an auxiliary heating coil. The fractions of the quantities of heat evolved by the halves of the wire, which are effective in producing the convection currents, will not in general be the same.

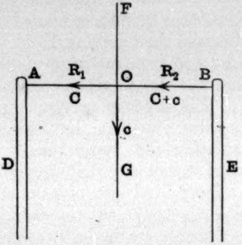

FIG. 2.

Let k_1 and k_2 be these fractions for the halves OA and OB respectively.

Let R_1 and R_2 be the resistances of the halves OA and OB respectively.

Let q be the rate of generation of heat by auxiliary coil, plus that which corresponds to small deflection of vane.

Then $$k_1C^2R_1 - k_2C^2R_2 = q.$$

A strong direct current, C, is then passed through the wire to raise its temperature to about 100 deg. The cooling in one half of the wire due to the Thomson effect is compensated by passing a small current, c, in the opposite direction through the other half and the lead OG. The reverse holds for any heating due to Thomson effect. An approximate calculation shows that, by suitably dimensioning the wires, the current c can be made as small as 0·002C. Thus the Thomson and Joule effects in the lead OG can be neglected.

On changing from alternating to direct current the resistances will, owing to the Thomson effects, undergo small variations. In the following treatment, for the sake of clearness, it is assumed that the figure applies to the case where the Thomson effect is positive.

Let σ = Mean Thomson coefficient in volts per degree,
C = Current through BA,
c = Compensating current through OB,
t = Difference of temperature between the centre O and the leads D and E as determined with thermo-junction at O and others on leads D and E,
δR_1 and δR_2 = Variations of the resistances R_1 and R_2 respectively.

Let the compensating current c be adjusted to give the same deflection as formerly. Then

$$k_1\{C^2(R_1+\delta R'_1)+\sigma Ct\}-k_2\{(C+c_1)^2(R_2+\delta R'_2) -\sigma(C+c_1)t\}=q. \quad (1)$$

The assumption that k_1 and k_2 for the Joulean heat are the same as for the Thomson heat is justifiable in the case where (1) there is a steep temperature gradient from the centre towards the end of wire AB, and (2) the greater part of the heat from wire AB is dissipated within the radiometer. Both these conditions can be nearly fulfilled.

Let C be reversed and compensation effected by adjusting c from B to O. Then

$$k_1\{C^2(R_1-\delta R''_1)-\sigma Ct\}-k_2\{(C-c_2)^2(R_2-\delta R''_2) +\sigma(C-c_2)t\}=q. \quad (2)$$

There remains on subtracting (2) from (1) and neglecting small terms

$$k_1C^2\delta R_1-k_2C^2\delta R_2+2k_1\sigma Ct+k_2\sigma t\{(C+c_1) +(C-c_2)\}-2k_2C(c_1+c_2)R_2=0. \quad (3)$$

where $\delta R_1=\delta R'_1+\delta R''_1$ and $\delta R_2=\delta R'_2+\delta R''_2$.

Let the compensating current c be now passed through the other half OA of the wire AB. Then similarly

$$k_2C^2\delta R'''_2-k_1C^2\delta R'''_1+2k_2\sigma Ct+k_1\sigma t\{(C+c_3)$$
$$+(C-c_4)\}-2k_1C(c_3+c_4)R_1=0. \quad (4)$$

It may be assumed that $\delta R_1-\delta R'''_1=\delta R_2-\delta R'''_2$. Hence addition of (3) and (4) and division all through by C gives

$$4\sigma t(k_1+k_2)-\frac{(c_1+c_2+c_3+c_4)}{2}(R_1+R_2)(k_2+k_3)=0. \quad (5)$$

To obtain this equation it has been assumed that (1) (c_3-c_4) and (c_1-c_2) may be neglected in comparison with C, and (2) k_1 very nearly equals k_2, and (3) R_1 very nearly equals R_2.

Thus $$\sigma=\frac{(c_1+c_2+c_3+c_4)(R_2+R_1)}{8t} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To measure t two additional thermo-junctions would be attached to leads D and E, preferably near the cover plate of the radiometer. The resistance (R_2+R_1) would be the total resistance of wire AB, and the parts of the leads D and E between the extreme thermo-junctions. This resistance could be measured with sufficient accuracy by the aid of a potentiometer or a voltmeter.

All the quantities on the right of equation (6) are measurable to a sufficient degree of accuracy for the determination of the Thomson coefficient. It is very difficult to separate the Thomson effect in a metal from other parasitic E.M.F.s arising from possible inequalities in the composition and crystallographic structure of the metal. The latter is more especially true of those metals in which the Peltier and Thomson E.M.F.s vary with direction through the characteristic crystal of the metal. Hence in many cases too much reliance cannot be placed on the final result.

SOUTH-WESTERN POLYTECHNIC, CHELSEA, S.W.

ABSTRACT.

The first part of the Paper relates to improvements which have been made in order to convert the original Joule convection apparatus into an instrument for the exact measurement of small steady rates of evolution or absorption of heat. These improvements consisted in (1) replacing the badly conducting glass enclosure and cardboard partition by others made of brass and copper respectively; (2) replacing the uncertain and variable magnetic control of the movement of the vane in Joule's apparatus by the elastic control of a quartz fibre; (3) shaping the channels, in which the vanes moved, so that the angular deflection of the vanes was proportional to the rate of evolution of heat; (4) reducing the size, so that a more uniform

temperature of its various parts could be easily maintained by (5) placing the radiometer within a concentric brass tube to exclude all extraneous heat excepting that which might be directed through apertures in its side towards the radiometer.

The sensibility of the instrument was measured by passing a current through a resistance loop in one of the compartments of the partitioned tube, and found to be equal to 0·52 mm. per microwatt, as measured on a scale at a distance of 1 metre from the mirror. Thus the instrument may be used for the measurement of feeble oscillating currents, it being about as rapid as a Duddell milliammeter.

To convert the apparatus into an instrument for the measurement of radiant heat it is suggested that the radiant heat be directed through a small rock salt or fluorite window in the side of a compartment on to a thin blackened metal disc supported centrally by a badly conducting fibre within the compartment.

Its use for the quick measurement of the heat given out by radium is also suggested.

It is suggested that small steady rates of evolution or absorption of heat might be measured by the compensation methods of Callendar or Ångström.

The second part of the Paper relates to a suggested method of measuring the Thomson effect with this radiometer. The method hinges on an experiment described by the author in "Nature," May 18, 1911, p. 380. In that apparatus the halves of a thin wire on either side of the partition are heated by the passage of an alternating current through thicker leads of the same metal. The Joule effects are compensated very nearly by an electro-deposit of the same metal, by scraping the thin wire, or by an auxiliary heating coil. The substitution of a direct current for the alternating current causes a slight heating in one and a cooling in the other half of the wire. The heating or cooling due to the Thomson effect in one half of the wire is compensated by passing a small measured current in the proper direction through the other half of the wire. This small current is passed through a thin lead of the same metal attached to the centre of the thin wire, and may be adjusted in four different ways. The temperature difference between the centre of the wire and the thick leads is measured with suitably attached thermojunctions.

The Thomson coefficient is expressible in terms of measurable quantities, and is equal to the product of the mean compensating current and the mean resistance of the halves of the wire divided by the temperature difference between the centre of the thin wire and its thick leads.

DISCUSSION.

Dr. W. H. Eccles stated that he had worked a good deal with other forms of convection instruments. The better-known type consisted of a helix of wire which was caused to rotate by the draught up the tube. Forbes in 1890 patented a convection galvanometer with a screw propeller placed in the draught tube over the heater. He had developed this by using a fine paper screw propeller suspended by a quartz fibre, and used it for measuring small oscillatory currents, though his old instrument was 50 times less sensitive than Mr. Jordan's.

The Author stated that Crookes in 1887 had used vanes set at 45° in a tube to measure convection currents.

VI. *Note on the Attainment of a Steady State when Heat Diffuses along a Moving Cylinder.* By MISS A. SOMERS, *B.A.*

COMMUNICATED BY DR. A. GRIFFITHS, BIRKBECK COLLEGE.

RECEIVED OCTOBER 8, 1912. READ NOVEMBER 8, 1912.

IN experiments aiming at the determination of thermal conductivity, carried out by Mr. Nettleton, and described in the "Proceedings of the Physical Society of London," Vol. XXII., April, 1910, a column of mercury having its ends at fixed temperatures is kept in steady longitudinal motion, and it is necessary to ascertain the time of attainment of a steady flow of heat along the column. This has been done by experiment, but the following suggests a method of calculating the time from purely theoretical considerations :—

Let K = the thermal conductivity of the material of the column,

v = the velocity of the column when moving from the cooler to the hotter region,

ρ = the density of the material of the column,

s = the specific heat of the same,

θ = the temperature above the enclosure at any point within the column of distance x from the plane normal to the column through its cooler end, at a time t measured from the beginning of the experiment,

L = the length of the column between the points at fixed temperatures,

θ_L = the fixed temperature at the hotter end,

θ_0 = the fixed temperature at the cooler end.

Let the initial temperature of the column be that of the enclosure.

Then, when the flow is from the cooler region, the equation which gives the temperature at any point within the column is

$$K\frac{d^2\theta}{dx^2} - v\rho s\frac{d\theta}{dx} = \rho s\frac{d\theta}{dt}.$$

The solution of this equation to satisfy the conditions of the experiment is

$$\theta-\theta_0=(\theta_L-\theta_0)\frac{1-e^{v\rho x/K}}{1-e^{v\rho s L/K}}+\Sigma A_i e^{v\rho x/2K}\sin\frac{i\pi x}{L}\cdot e^{-\left(\frac{v^2\rho s}{4K}+\frac{i^2\pi^2 K}{\rho s L^2}\right)t},$$

where i has every positive integral value in turn, and

$$Ai=\frac{2i\pi}{L^2\left(\frac{v^2\rho^2 s^2}{4K^2}+\frac{i^2\pi^2}{L^2}\right)}\left\{(-1)^i\theta_L e^{-\frac{v\rho s L}{2K}}-\theta_0\right\}.$$

As the rate of flow of heat past any point in the cylinder is given by

$$K\frac{d\theta}{dx}-v\rho s(\theta-\theta_0),$$

it follows that the rate of flow across the hotter end of the cylinder is given by

$$(\theta_L-\theta_0)\cdot\frac{v\rho s}{e^{v\rho s L/K}-1}$$

$$+\frac{2K}{L}\Sigma\frac{i^2\pi^2 K}{\frac{v^2\rho^2 s^2 L^2}{4K}+i^2\pi^2 K}\ \theta_L-(-1)^i\theta_0 e^{v\rho s L/2K}\}e^{-\left(\frac{v^2\rho^2 s^2}{4K}+\frac{i^2\pi^2 K}{L^2}\right)\frac{t}{\rho s}}$$

being $(\theta_L-\theta_0)\cdot\frac{v\rho s}{e^{v\rho s L/K}-1}$ when the steady state is attained.

Thus the formula gives the time of attainment of the steady state.*

It may be noted that a strictly analogous equation is applicable to the case of the diffusion of a salt in solution through a tube, both ends of which are kept at constant concentration, as described by Mr. Clack in the "Proceedings of the Physical Society of London," Vol. XXI., 1908, and Vol. XXIV., December, 1911. He has calculated the time required for the attainment of a steady rate of diffusion when the liquid is at rest, and concludes from experimental evidence that the actual small variable velocity of the liquid will not appreciably affect this time.

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* Calculations based on these formulæ show that, with velocities of the magnitude suggested by Mr. Nettleton, an approximately steady state, involving less than 0·5° difference from the ultimate temperatures along the column, will be attained in just over two hours.

Assuming that the velocity is uniform, this conclusion is supported by calculation from a formula similar to the above, since the velocity in this case is of far lower order than the coefficient of diffusion.

ABSTRACT.

The Paper dealt with the case of a column of mercury moving with uniform speed between two fixed temperature sources. The differential equation for the temperature within the column was stated and its solution given, and it was shown how the time of attainment of a steady state could be obtained from the latter. The case of the diffusion of a salt in solution up a tube could be treated in the same manner.

DISCUSSION.

Dr. A. **Griffiths** asked if some Fellow would solve the problem when the velocity, instead of being constant, was a periodic function of the time.

Mr. B. W. **Clack** stated that Miss Somers had referred to his work on diffusion, and in such a slow phenomenon it was important to save as much time as possible. The velocity of the liquid down the diffusion tube referred to was in his experiments natural and not artificial, depending on the change in volume of the solution as it became less concentrated by the diffusion. This velocity was very slow. In his apparatus it was of the order 1 cm. in four months, and he felt justified in assuming that this would not materially alter the time required to attain the steady state. Experiments showed that this assumption was legitimate.

Mr. R. **Appleyard** drew some analogies between the differential equation used and that for the flow of electricity along conductors.

VII. *The Thermomagnetic Study of Steel.* *By* S. W. J. Smith, *M.A., D.Sc., Assistant Professor of Physics, Imperial College of Science.*

Received October 21, 1912. Read November 8, 1912.

Among the phenomena exhibited by magnetic materials, one recurs so frequently that it may reasonably be looked for in any substance whose magnetic properties are still incompletely known.

This phenomenon appears in curves plotted to show corresponding values of the permeability μ and the temperature θ for different values of **H**, the magnetising field. For each particular field there is in general a temperature at which the permeability is a maximum. When the field strength is comparatively large the maximum is not very pronounced and the $\mu\theta$ curve is concave to the axis of θ over a wide range. As the field strength is reduced, however, the maximum becomes (within certain limits) more and more clearly defined. The range over which the curve is concave to the axis of temperature becomes ultimately only a few degrees. Simultaneously the temperature of maximum permeability approaches the critical temperature of the substance.

These characteristic variations constitute the phenomenon referred to above. They suggest a method of exhibiting, with the maximum of clearness, the co-existence of different magnetic constituents in the same material. The object of this communication is to illustrate this method by showing how strikingly the presence of carbide of iron in steel lends itself to demonstration by its aid.

It has been shown in an earlier Paper* that the permeability of this carbide is relatively very small above 230°C. The measurements were therefore restricted to temperatures between that of the air and 250°C. They were made by the ballistic method upon a sample of the nearly pure carbon steel kindly given to me by Prof. Arnold, and stated to contain the following percentages of elements other than iron: $C=0{\cdot}85$, $Si=0{\cdot}05$, $Mn=0{\cdot}06$, $S=0{\cdot}03$ and $P=0{\cdot}02$. The material was supplied in the form of carefully annealed rods half an inch in diameter. From one of these a tube 5 cm. long was constructed, having an inductive cross-sectional area of nearly 1 sq. cm.

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* "Proc." Phys. Soc., Vol. XXIV., 1911, p. 64.

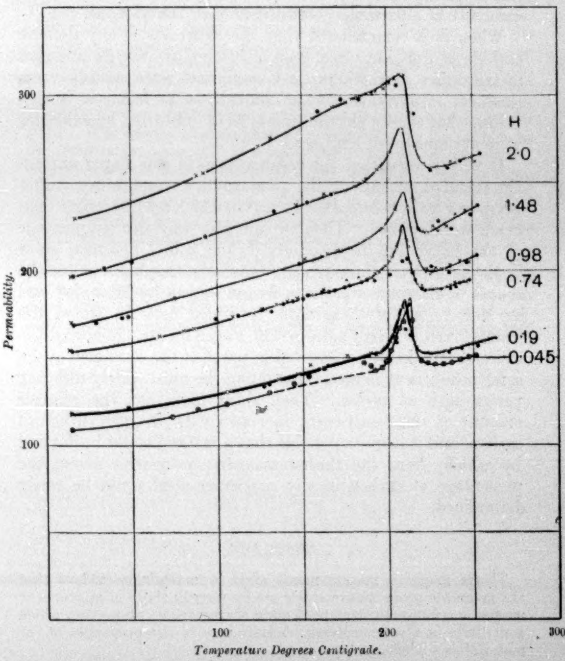
It was found by trial that cotton immersed in melted paraffin wax retains its insulating power sufficiently well, for the purposes of magnetic measurements, up to 250°C., although it chars and becomes defective at much lower temperatures when heated in air. The primary and secondary coils were therefore constructed of cotton-covered wire and the "ring" was suspended, by the leads from the coils, within a test tube containing more than enough wax to cover it completely. Heat was conveyed to the test tube through a thick containing tube made of copper. The temperature of the ring was measured by means of a calibrated copper-constantan couple placed within it and also by means of a mercury thermometer in contact with its outer surface.

Some of the results of the experiments are shown in the figure on page 79 in which the abscissæ are temperatures and the ordinates are the approximate permeabilities, in absolute units, corresponding with the values of **H** marked opposite the curves. The points denoted by crosses represent observations taken during cooling. Only a few observations (represented by dots) were taking during heating, owing to the relatively greater difficulty of temperature regulation in that case.

The curves present various features of interest which it seems inadvisable to discuss in detail until further experiments upon this and upon other steels containing different percentages of carbon have been completed.

It seems obvious, however, that the contribution of the carbide towards the apparent permeability of the material as a whole becomes more and more restricted in its range as the strength of the magnetising field is lowered. Until, when the field is below 0·2 gauss, the even sweep of the curve between 20°C. and 250°C. suffers no considerable disturbance except over some tens of degrees in the neighbourhood of the critical temperature of the carbide. To show this the more clearly in the figure the curve for $\mathsf{H}=0{\cdot}19$ C.G.S. is drawn more boldly than the rest. Attention may also be drawn to the curve for $\mathsf{H}=0{\cdot}045$, the weakest field in which measurements could conveniently be taken with the apparatus used. In this case the maximum permeability occurs in the neighbourhood of 213°C. and is nearer the critical temperature than any of the other maxima. The $\mu\theta$ curve also descends more steeply beyond the maximum than in any of the other cases. Several of the curves—this one in particular—show evidence of a secondary maximum which may possibly be due to some unsuspected constituent.

It is not, of course, to be supposed that the fields given in the figure are the actual fields in which, for example, a ring of the



pure carbide would exhibit variations of permeability precisely similar to those just described. The carbide, as we have examined it, is embedded in a material of which the permeability is in general very different from its own. The effective field within it will therefore be different in general from the average field within the steel as a whole and will, moreover, be different in different parts of the ring—unless the distribution of the carbide is exceptionally regular.

It is, however, unnecessary for the present purpose to lay stress upon this point. The immediate object is only to show that, by experiment, fields can be found which make the existence of a particular constituent very conspicuous.

When it is remembered that the existence of any definite carbide of iron has often been a subject of dispute amongst metallurgists and others, and even now rests mainly upon chemical analyses somewhat difficult to perform, it is surprising that purely physical methods of indicating its existence were not multiplied long ago.

It will be seen that the results given in this Paper explain the rounded outlines of the permeability-temperature curves given in Fig. 3 of the earlier Paper* in which fields stronger than **H**=2 were applied. They explain also why the temperature of the maximum permeability below 250°C. becomes lower as the field strength is raised. Similarly, they account for the shapes of the curves given in recent Papers by Maurain† and by Moir.‡ All of these data, to which may be added the results given in a still more recent Paper by Mr. Guild and the writer,§ can therefore be cited in proof of the existence of the same constituent in steels containing the most widely differing percentages of carbon. They show also that the relative amount of this constituent increases with the percentage of carbon, and it is quite possible that a method could be devised by which, from the thermo-magnetic properties alone, the percentage of carbon in any particular steel could be easily determined.

ABSTRACT.

Thermomagnetic measurements make it increasingly evident that the magnetic properties of steels are frequently those of mixtures of magnetic substances, each possessing characteristic properties, which contribute in a comparatively definite way to the properties of the material as a whole.

In the case of a simple ferromagnetic substance, magnetising fields can generally be found in which the permeability variation with tem-

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* *Loc. cit.*, p. 66.

† "Ann. de Chim. et de Phys.," Vol. XX., 1910, pp. 353–389.

‡ "Proc." R.S.E., Vol. XXXI., 1911, pp. 505–516.

§ "Proc." Phys. Soc., Vol. XXIV., 1912, pp. 342–348.

perature is comparatively small except in the neighbourhood of the critical temperature. In such fields there is a very clearly marked peak in the permeability temperature curve for the substance. The explanation of this peak which the molecular theory affords is well known, and suggests that the phenomenon should be found common to all ferromagnetic substances. The immediate object of the present Paper is to show that it is exhibited by the carbide of iron (cementite) which exists in annealed carbon steels. For this purpose it is not necessary to isolate the carbide because, as shown in the Paper, the phenomenon is quite clearly discernible in the permeability temperature curves for the steel. The particular steel examined contained 0·85 per cent. of carbon. It was found that the fields necessary to evoke the comparatively sudden variations in the permeability of the carbide above described are small and such that the permeability variation of the iron present along with the carbide is slight in the neighbourhood of the critical temperature of the latter. The sudden gain and loss of permeability by the carbide as the temperature alters will be roughly equivalent to sudden removal and replacement of gaps in the magnetic circuit through the steel. They should therefore be attended by correspondingly sudden rise and fall of the apparent permeability of the material as a whole. This is found to be the case. There is a sharply marked peak near 210°C. upon the permeability temperature curve for the steel.

In the absence of measurements between 200°C. and 220°C. the peak would escape notice and it is for this reason, probably, that it has not been recorded before. It could scarcely be found by accident. The search for it was prompted by the considerations outlined above in conjunction with results obtained in earlier work with Messrs. White and Barker.

VIII. *The Law of Plastic Flow of a Ductile Material and the Phenomena of Elastic and Plastic Strains.* By CHARLES EDWARD LARARD.

RECEIVED NOVEMBER 1, 1912. READ NOVEMBER 22, 1912.

IN a Paper recently read before Section G of the British Association,* the author, *en passant*, stated briefly certain conclusions he had arrived, at connecting the variables, torque, time and twist, in the case of a ductile material which had been twisted to destruction. In the present Paper he offers a sufficiently full account of the investigation on a piece of mild steel of the form and dimensions shown in Fig. 1, which was twisted to destruction under a pure torque of increasing magnitude.

It is a matter of common experience to those who have to deal experimentally with the straining of material to destruction that the load, and, in the case of torsion, the twisting moment, must be increased more and more slowly, or at a diminishing rate of increase with respect to time, as the test proceeds, and that towards the end of the test, when the material is reaching its breakdown, very small increments in the load or torque are accompanied by very large amounts of plastic or permanent strain.

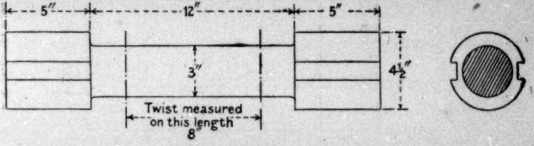

FIG. 1.

Three principal cases presented themselves to the author before commencing the experimental work. These are the cases where—

1. The angular velocity of straining or twisting is maintained constant, the torque being increased continuously so as to maintain equilibrium between the twisting moment and the moment of resistance of the specimen.

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* Dundee Meeting, September, 1912.

2. The torque-time rate (or, as it may be termed, the velocity of the torque) is kept constant and the twist-time rate being continuously increased to maintain equilibrium.

3. After each step-by-step increment in the torque time is allowed for the greater part of the plastic strain to be recorded before the next increase in the magnitude of the torque.

The author has made many experiments in connection with all three, but he here presents a partial solution of the first

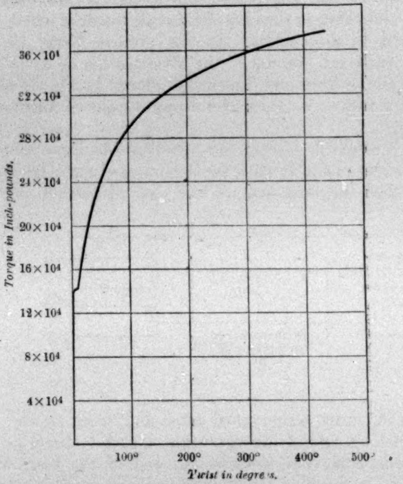

Fig. 2.—Autographic Torque Twist Diagram (for 12 in. Length between Shoulders.)

problem only, in the particular case where the specimen was twisted to destruction at a uniform angular velocity, ω, of 0·1145 deg. per second. For the purpose of this and other tests three special instruments or pieces of apparatus were used :—

1. A recorder to draw autographically throughout the test the torque-twist curve for the full 12 in. length of specimen between the shoulders. This curve is reproduced in Fig. 2,

and shows clearly the kind of curve obtained with a yield point very clearly defined.

2. A torsion indicator with a large graduated aluminium circle for determining the angle of twist, at any stage in the test, on an 8 in. length of the specimen, being the distance between the gauge points.

3. Apparatus for simultaneously marking the time on the torsion circle, the steelyard scale and the autographic diagram, so that both the torque-time and twist-time rates as well as the corresponding accelerations could be deduced.

The material for the purpose of these tests was supplied by Messrs. Firth, of Sheffield. The specimen was forged at an approximate temperature of 850°C. as a cylindrical piece 4¾ in. diameter from billets of Siemens-Martin mild steel 1¾ in. round. It was then annealed in an oven for one hour at a temperature of 750°C., after which is was removed, buried in sand, and allowed to cool gradually. After cooling it was machined to the dimensions given in Fig. 1.

A chemical analysis of drillings taken from the specimen gave the following composition :—

Carbon	0·320 per cent.
Silicon	0·160 ,, ,,
Manganese	0·530 ,, ,,
Sulphur	0·028 ,, ,,
Phosphorus	0·030 ,, ,,

The cost of the steel and its machining was defrayed out of a Government grant received through the Royal Society.

The twisting of the specimen was carried out in the large testing machine* installed in the author's laboratory at the Northampton Institute, London.

It will be sufficient for the purpose of this Paper to state briefly the manner of holding and twisting the specimen and applying the torque. One enlarged end of the torsion specimen is inserted into a die-block sliding into a bracket fixed to the steelyard lever of the testing machine, while the other enlarged end is held by a die which slides into the boss of a large circumferentially toothed straining wheel connected by gearing to an electric motor. The two end dies and attachments ensure that the geometrical axis of the specimen when produced shall be coincident with the line of contact of the knife edge of the steelyard lever with its support. The rotation of the large wheel effected the twisting of the specimen. To keep the steelyard lever horizontal so that there should be equilibrium between the torque and the resisting moment, the poise weight was moved forward in the positive direction, defining by its position on the steelyard scale the magnitude of the torque.

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* For descriptions *see* the author's Papers in the "Proceedings" of the Institution of Mechanical Engineers, Vols. III. and IV., 1907, and the British Association Paper published in "Engineering," on Sept. 3 and 10, 1909.

Before starting the test, the specimen was placed in the machine and the vernier scale attached to the poise weight adjusted to zero on the steelyard scale, with the beam floating horizontally in its neutral position. At a given moment, the time being noted, the electric motor was started and kept running throughout the test, driving the gear and twisting the specimen at the uniform rate stated. The poise weight was advanced continuously so as to keep the lever (after the yield period was passed) in its horizontal position throughout the test, and time intervals were marked simultaneously at predetermined times on the steelyard scale, the torsion circle and the autographic diagram. For the purpose of receiving the time marks, a specially prepared paper was used, glued by fish glue both to the aluminium torsion circle and to the steelyard scale on the lever. The pencils for marking time were

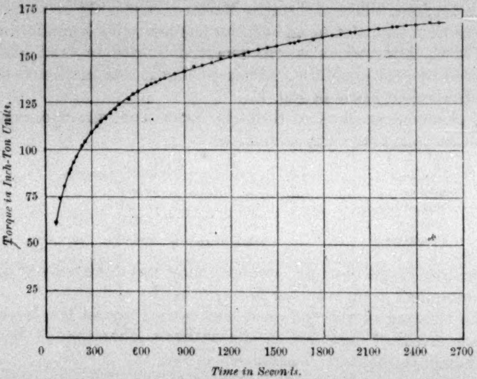

Fig. 3.—Torque Twist Curve for Steel Specimen, 3 in. Diameter by 8 in. Long.

pieces of pointed brass wire which, on the depression of a key at required intervals, came into contact with the paper.

During the testing to destruction of this specimen some hundreds of observations of time, torque and twist were made. To publish all the tabular data would make a very bulky Paper. The author has deemed it advisable, therefore, after reducing the results, to give his deductions in graphical and mathematical form.

Some of the plottings for the torque, T time t, and the twist θ for the 8 in. length are given in Fig. 3, and the curve obtained gives a good idea of the kind of variation between these two variables. The experimental values of $\frac{\Delta t}{\Delta \mathrm{T}}$ obtained from the tabular data have been plotted as ordinates to a time base (*see* Fig. 4) and an inspection of the result indicates that there is a linear relationship between these two variables with, in the limit, assuming the line to pass through the origin, an equation of the form

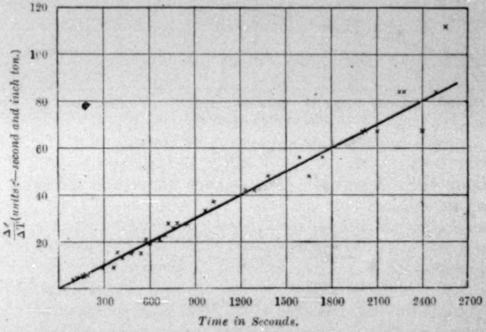

FIG. 4.

$$\frac{dt}{d\mathrm{T}} = bt \quad \text{or} \quad \frac{d\mathrm{T}}{dt} = \frac{1}{bt}.$$

Consequently $\frac{d\mathrm{T}}{dt} \times t = \frac{1}{b} =$ constant, *i.e., the rate of increase of the torque with the time varies inversley as the time.*

This is at once an indication that the two variables, torque and time, are connected by the compound interest law, so that we may write

$$t=a\varepsilon^{b\mathrm{T}}.$$

If $\frac{\theta}{\omega}$ is substituted for t the above equation becomes

$$\theta=a\,\omega\varepsilon^{b\mathrm{T}}.$$

It should be noted that the line in Fig. 4 has been drawn with an equation, $\frac{dt}{d\mathrm{T}}=\frac{t}{30}$, obtained after the value of b had been found from the plottings in Fig. 5.

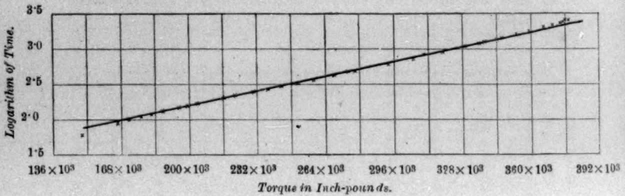

Fig. 5.—Torque and Logarithm of Time.

Taking the differential equation given above and differentiating we obtain

$$\frac{d^2\mathrm{T}}{dt^2}=-\frac{1}{bt^2}$$

or

$$\frac{d^2\mathrm{T}}{dt^2}\times t^2=\text{constant},$$

i.e., the acceleration which is negative, or, as it may be called, the deceleration of the torque with respect to time varies inversely as the square of the time.

Since $t=a\varepsilon^{b\mathrm{T}}$ represents to a first approximation the relationship between the variables T and t we have

$$\log_{10} t=\log_{10} a+b\mathrm{T}\log_{10}\varepsilon,$$

which is a straight line.

Plotting the experimental values of $\log_{10} t$ and T as in Fig. 5, further confirmation of the truth of the fundamental equation is obtained to at least a first degree of approximation. The values of the constants a and b are also obtained from these plottings, giving

$$t=8\cdot23\varepsilon^{\frac{T}{29}}=8\cdot23(1\cdot035)^{T},$$

where t is in seconds and T in inch-ton units.

Results similar in form to the foregoing may be stated where θ is the independent variable, since $\theta=\omega t$.

Having deduced the relationships given as representing very closely the results of the experiments, the author wishes to call attention to the plottings of the logarithms of the time near the two axes in Fig. 5. It will be noticed that for a short range of torque the points obtained lie a little below the line drawn suggesting that there may be a small time constant t_0 in the exponential equation.

Making this assumption the equation would then take the form

$$t+t_0=a\varepsilon^{bT},$$

and this equation is the general solution of the differential equation

$$\frac{dt}{dT}=bt+\text{constant},$$

suggesting that the line of Fig. 4 may not strictly pass througn the origin although it is close to it. On trial it was found that if t_0 was taken at about 6 seconds and the plottings for Fig. 5 were made between $\log_1\ (t+6)$ and T the points were brought up to the line given, determining another form of the general equation. This constant t_0 makes no appreciable difference in the value of b.

Owing to the small value of the constant t_0 the other plottings for $\log_{10}t$ are not appreciably affected, the differences in the values of $\log_{10}(t+6)$ as compared with the values of $\log_{10}t$ become negligible within a very short range of the torque. In this connection it should be stated that the duration of twisting was 42 minutes.

The author wishes here to summarise certain other conclusions he has formed as a result of many experiments extending over five years, reserving the full account for later publication.

1. *The Elastic Period.*

(a) Plastic strain is always produced by applied stress, and when its amount becomes large enough to be detected by the strain instrument the actual linearity between stress and elastic strain is no longer directly obvious.

(*b*) The *recorded* limit of elasticity depends on the degree of accuracy to which the strain instrument is capable of indicating the strain, *i.e.*, there is no definite limit of elasticity, the recorded limit being merely an instrument limit.

(*c*) In the case of a tension test where the increase of strain is measured in the direction of the applied stress, linearity between stress and strain is apparently very pronounced, but in the case of a torsion test very small increases in the elongation of helically twisted generating lines of the specimen are accompanied by the relatively large angles of torsion measured by the twist-strain instrument, with the result that plastic strain is observed very early in the test giving low instrument limits of elasticity.*

(*d*) There is elastic strain, whether it is obscured by plastic flow or not, which is closely proportional to the torque (or load) for all values of the torque (or load) up to the maximum, and it is given by $\theta_e/\mathrm{T}=\theta_1/\mathrm{T}_1$, where T is any torque not exceeding the maximum load, and T_1 and θ_1 are from observations of torque and twist respectively during the elastic period of instrumental linearity between stress and strain. Consequently the usual formula may be used for calculating elastic strains for values of the torque up to near the maximum.†

(*e*) The total strains produced under applied stress, even before the yield period is reached, depend on the time as well as on the torque (or load).

(*f*) The linearity between elastic strain and applied stress may be rendered obvious up to a high value of the load by suppressing the plastic flow.‡

2. *The Yield Period.*

(*a*) The plastic strain phenomena under constant load during this period depends on the time. If the load is imposed on a specimen either *very* slowly or *very* quickly§ there is no yield period, the former case corresponding very closely to isothermal straining and the latter to adibatic.

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* *See* Author's Paper, "Engineering," Sept. 3 and 10, 1909; also Author's Paper, "Proceedings" of the Institute of Automobile Engineers, Jan., 1911.

† *See* Paper by the Author, "Minutes of Proceedings" Institute of Civil Engineers, CLXXIX., 1909.

‡ Author's Paper, Institute Civil Engineers, 1909.

§ "Strength of Materials," Ewing, pp. 41-42, 1899 Ed.

(*b*) *Most General Case.*—When the yield torque (or load) is reached and where equilibrium is maintained between the load and the resistance, the velocity of the plastic strain is rapidly and increasingly accelerated for a short period of time due to some rapidly and increasingly softening process going on in the material, the acceleration $\ddot{\theta}$ rapidly reaching a maximum value followed for a short period by a rapidly decreasing acceleration, the velocity $\dot{\theta}$ still increasing until the acceleration reaches zero value with momentary uniform velocity. From this point and for by far the greater part of the yield, the velocity $\dot{\theta}$ undergoes constantly decreasing retardation due to some gradual and increasingly hardening process going on in the material, until the curve in $\ddot{\theta}$ and t becomes, or tends to become, asymptotic to the axis of time with the limit zero deceleration and a uniform value of $\dot{\theta}$.

(*c*) From what has been stated above it follows that the yield period is produced when from the conditions of loading acceleration of plastic flow is produced and further that where the torque-time and torque-twist ratios are kept below certain critical values, and, therefore, where there is no acceleration but only a retardation, no yield period is produced, and the elastic-torque-twist line and the plastic-torque-twist curves are compounded into a smooth and continuous resultant torque-twist diagram.

(*d*) The yield period may be due to the production of a fluid state (liquefaction) in parts of the material under the acceleration with subsequent relegation and hardening producing deceleration after the parts have adjusted themselves to their new positions. The torque-twist curve which is continuous denotes a condition of semi-plastic flow.

(*e*) The yield period which takes place under the circumstances indicated can be raised by strain* and heat treatment at low temperatures to almost any position with respect to a torque-twist curve.

3. *Total Strain.*

(*a*) If the above results are accepted the total strain for any value of the load is given by

$$\underset{\text{total}}{\theta} = \underset{\text{elastic}}{\theta_e} + \underset{\text{plastic}}{\theta_p},$$

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* J. Muir: "Transactions" Royal Society, Vol. CXCIII. (1900), and Author's Paper "Minutes of Proceedings" Institute of Civil Engineers, 1909.

Where the elastic strain is a linear function of the load for all ordinary speeds of loading possible with a statical testing machine, and where the plastic strain is function of both load and time.

ABSTRACT.

The author gave an account of the twisting to destruction at a uniform angular velocity of a cylindrical steel specimen 3 in. diameter and of his deductions from the experimental data. The following deductions were made :—

1. The rate of increase of the torque with the time varies inversely as the time.

2. The acceleration of the torque velocity which is negative or, as it may be called, the de-celeration, varies therefore inversely as the square of the time.

3. The variables, time t, and torque T, are connected by the compound interest law.

More exactly $t+t_0=a\epsilon^{bT}$, where t_0 is a time constant. Corresponding results in terms of the angle of torsion θ and T obviously followed, since $\theta=\omega t$, where ω is the angular rate of straining.

The author also summarised certain other conclusions he has formed as a result of many experiments extending over five years, illustrating his arguments by means of original diagrams, but reserving the full account for later publication.

DISCUSSION.

Sir R. **Hadfield** considered the Paper an important one. With respect to the question of elastic limit, many years ago he was working on manganese steel, which has a very low elastic limit but a very high tenacity, yet it is used for just those purposes [for which high elastic steel is used. In carrying out some experiments on manganese steel 25 years ago, by repeated re-toughening the specimen he elongated an 8 in. bar to 14 in. He also drew attention to the behaviour of steel when the rate of stress was exceedingly great, such as in the Admiralty tests with projectiles. A 14 in. shot with 35,000 ft.-tons of energy will pierce a 12 in. plate in $\frac{1}{100}$ second. So we get here problems not met with in ordinary engineering, yet records taken of the passage of the shot show that the steel behaves very much the same under these conditions as it does in an ordinary testing machine.

Dr. C. **Chree** pointed out that there were two ways in which the elastic limit could be defined : (1) Either as the range within which Hooke's law holds or (2) as the range within which the strain reduces to zero again when the stress is removed. He thought the results would have been clearer from a physical standpoint if the author had employed t (time) throughout as the independent variable in his mathematical expressions. If he had taken dT/dt as ordinate and t as abscissa in Fig. 4 his theoretical curve would have been a rectangular hyperbola, which would have served satisfactorily to show the agreement between the observations and the theory. The real physical significance of the experiments seemed to be that under the conditions of test $\theta dT/dt$ was a constant. Had the author tried whether this relation held when θ did not increase uniformly with the time ? The formula required some restriction as it made dT/dt initially infinite.

Mr. F. L. **Hopwood**, commenting on Mr. Larard's view that the elastic yield persists during the plastic yield, mentioned that a similar conclusion had been drawn by Andrade (" Proc." Roy. Soc., A., Vol. 84, 1910) from tension experiments on lead and copper.

Prof. **Margetson** pointed out that since the rate of strain was constant, Figs. 2 and 3 were practically similar, and he suggested that if the results had been stated in terms of θ and T, instead of between t and T, the results would have been more in accordance with the author's previous statement * of the compound interest law in the form $a\epsilon^{bT} = \theta$.

Mr. R. **Appleyard**, in drawing attention to Fig. 5, remarked:—There is a certain amount of divergence between the calculated firm line and the line passing through the plottings, and Mr. Larard suggests that to account for this it may be necessary to introduce a small time-constant t_0. It should be observed, however, that in working with logarithmic curves a small error in the constants is, as a rule, sufficient to produce a divergence of the kind here obtained. Hence it is worth while considering whether the divergence is due to the author having selected scarcely the best value for b. In Fig. 4 the observation points are somewhat scattered, and although it is convenient to draw a line from the origin to the point (2400, 80) which makes $b = \frac{1}{30}$, this is not necessarily the best line to fit the final result of Fig. 5. If the line of dots in Fig. 5 is continued backwards to cut a vertical line through the origin it will be found to give a value

$$\log_{10} a = 0\cdot64.$$

Hence, $a = 4\cdot37$, instead of the author's 8·23, would meet this case. It is, therefore, desirable to examine b very carefully before assuming the existence of a quantity t_0. The relation between T and t is, therefore, in this case

$$\log_{10} t = 0\cdot64 + bT \log_{10} \epsilon.$$

Now, take the best point in Fig. 5, and it is seen that there is agreement, where $T = 264 \times 10^3$, and $\log_{10} t = 2\cdot625$, as nearly as can be judged from the diagram. Hence, b can be derived from

$$2\cdot625 = 0\cdot64 + \frac{b \times 264 \times 10^3 \times 0\cdot4343}{2{,}240},$$

since T has here to be in inch-tons. This gives $b = 0\cdot0388$, or $\frac{1}{b} = 25\cdot8$. Now in Fig. 4 is co-ordinated $\frac{dt}{dT}$ and t, and since

$$\frac{dt}{dT} = bt,$$

we can select a value, say, $t = 2{,}400$, and we find

$$\frac{dt}{dT} = 0\cdot0388 \times 2{,}400 = 93\cdot07,$$

which is well above the 80 selected by the author. This point, if joined to the origin, gives a line which is within the observation points, and the effect on Fig. 5 would be that the firm line in that diagram would pass through the plottings without the introduction of a new factor t_0.

The **Author** replied, and supplemented his remarks as follows: The shot experiment quoted by Sir Robert Hadfield is suggestive of problems of great importance from a physical as well as from a practical point of view. If it be established as true in a more universal application that the stress-time rate multiplied by the strain is constant then it follows that if the velocity and energy of impact are great enough the small fraction of a second representing the duration of impact is insufficient to allow of the development of disintegrating strain in the bulk of the shot, and so penetration takes place. The Author agrees with Dr. Chree that the real physical significance of the experiment is that $\theta\frac{d\Gamma}{dt}$ is constant. This interpretation was definitely stated by him in the circulated abstract of his British Association Paper.* In further reply to Mr. Appleyard the author states that he has not made the mistake of allowing logarithmic differences to involve him in any serious error. The line given by him in Fig. 5 gives an intercept which is the log of 8·23. That the value $b=\frac{1}{30}$ is evidenced by the fact that when the different values of $d\Gamma/dt$ are multiplied by the *measured* angles of twist the products, throughout the entire range from the end of the yield to near the maximum load are astonishingly close to 3·435, this constant being derived as follows :—

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* "Proceedings" Institution of Mechanical Engineers, 1907.

Since
$$\frac{d\Gamma}{dt}=\frac{1}{bt}=\frac{\omega}{b\theta},$$

we get, on substituting the values $\omega=0{\cdot}1145$ and $b=\frac{1}{30}$

$$\theta\frac{d\Gamma}{dt}=3{\cdot}435.$$

Mr. Appleyard in drawing another line for Fig. 5 may not perhaps know that in all such diagrams there is a slight flexure downwards near the bottom of the line and a slight flexure upwards at the other extremity. The author is investigating the significance of this peculiarity. With respect to the interpretation of the constant t_0 first suggested, but afterwards qualified, the author has some feeling of uncertainty. It is not unlikely that its place should be taken by a very slowly increasing function of the torque corresponding to the *growth* of the elastic strain with the time. Replying further to Mr. Hopwood, the author holds the view, as may be gathered by this Paper and his previously published Paper,† that for steel not only does the elastic strain persist, but that it is continuously augmented under increasing load and time until it reaches its maximum value at some load not far below the maximum.

In reply to Mr. Hopwood the Author stated that if his results on "double and simultaneous flow" had been in any way anticipated, so much the better for the theory, but at the same time he called attention to the results in his previous Papers. Prof. Margetson's point was covered in the Paper referred to since the angular velocity of straining was there given.

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* "Report," British Association, 1912.

† "Proceedings" Institution of Civil Engineers, 1909.

IX. *The Effects of Holes and Semicircular Notches on the Distribution of Stress in Tension Members.* By E. G. Coker, *M.A., D.Sc., Professor of Mechanical Engineering in the City and Guilds of London Technical College, Finsbury.*

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Received in Revised Form November 12, 1912.

In a previous Paper* the author has determined approximately the difference of the principal stresses at the minimum sections of tension members having notches of various forms, and in the present Paper some experimental values are obtained for circular holes and semicircular notches in tension members, and these are compared with the results of calculations. The method of experiment adopted is similar to that described in a former Paper.† A strip of transparent xylonite is cut into shape and subjected to stress in any convenient way. The distribution of the stress is then examined by the optical effects produced when plane or circularly polarised light is passed through the plate at a perpendicular incidence.

The optical effect of the stress distribution at any point is to cause a retardation, R, in the component rays into which the incident beam is broken up, and to a first approximation R is proportional to the difference of the principal stresses $\widehat{\theta\theta}$ and $\widehat{rr}$ at a point r, θ, and to the thickness T of the plate, so that we have

$$R = (\widehat{\theta\theta} - \widehat{rr})CT$$

where C is a constant.

The retardation can be measured directly in terms of a wave-length scale, and the stress difference can be inferred therefrom, but in the present Paper the numerical values have in all cases been determined by reducing the field of view to darkness by using a stressed member placed along the direction of one of the principal axes of stress at the point considered, so that the stress difference is measured directly.

A case of primary importance is that afforded by a cylindrical hole in a tension member, such as occurs in many forms of engineering construction, although generally the conditions are complicated by the stress caused by the pressure of a rivet in the hole, and also by the stress due to the grip of the rivet heads. For plane stress the equations of equilibrium are satisfied when the radial stress rr, the tangential stress $\widehat{\theta\theta}$, and the shear stress $\widehat{r\theta}$ can be expressed in terms of Airy's function F, where

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* "Trans." Inst. Naval Architects, April, 1911.

† "The Optical Determination of Stress," by E. G. Coker, "Phil. Mag.," 1910.

$$\widehat{rr}=\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}+\frac{1}{r}\frac{\partial F}{\partial r},\quad \widehat{\theta\theta}=\frac{\partial^2 F}{\partial r^2},\quad \widehat{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right). \quad . \quad (1)$$

For the case of a hole in an infinite plate subjected to stress p it can be shown* that the function F has the value

$$F=\tfrac{1}{4}p\left\{r^2-2a^2\log r-\frac{(r^2-a^2)^2}{r^2}\cos 2\theta\right\}, \quad . \quad . \quad . \quad (2)$$

from which we readily obtain the stresses in the form

$$\widehat{rr}=\tfrac{1}{2}p\left\{1-\frac{a^2}{r^2}+\left(1-\frac{4a^2}{r^2}+\frac{3a^4}{r^4}\right)\cos 2\theta\right\}, \quad . \quad . \quad . \quad (3)$$

$$\widehat{\theta\theta}=\tfrac{1}{2}p\left\{1+\frac{a^2}{r^2}-\left(1+3\frac{a^4}{r^4}\right)\cos 2\theta\right\}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\widehat{r\theta}=\tfrac{1}{2}p\left\{-1-\frac{2a^2}{r^2}+\frac{3a^4}{r^4}\right\}\sin 2\theta, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where a is the radius of the hole.

The same result has been recently obtained by Suyehiro,† using the method of generalised plane stress. For a tension member with a small hole in it sufficiently far from the edge we see that the stress across the minimum transverse section rises slowly as we approach the discontinuity, and finally very rapidly. When $r=a$ and $\theta=\frac{\pi}{2}$, we have $\widehat{\theta\theta}=3p$ and $\widehat{rr}=\widehat{r\theta}=0$, or the maximum stress is three times the uniform stress at a distance from the discontinuity.

In order to experimentally determine the stress at the minimum section of a tension member when pierced by a central hole, strips of xylonite were prepared, 1 in. in width and 0·186 in. thick, and holes were cut in these ranging from $\frac{1}{16}$ in. to $\frac{1}{2}$ in. These specimens were stressed in a simple form of apparatus for applying definite tension loads.

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* Föppl, "Vorlesungen über Technische Mechanik," Vol. V., p. 352.

† "Engineering," September 1, 1911.

In the earlier experiments the holes were drilled through from one side of the plate in the ordinary fashion, but this was found to slightly damage the material, owing to the drill pushing the last section of uncut material outwards and causing a slight strain around the edge of the hole. In order to overcome this source of error, which had a considerable influence on the stress determinations, the holes were drilled from both sides, and rather smaller than the final size, the correct diameters being obtained by removing the remaining material with a fine file. This treatment caused no injury to the specimen and more accurate stress determinations became possible.

At first a total load of 200 lb. was applied to the specimens, but this was afterwards reduced to 100 lb. to minimise possible errors due to imperfect elasticity of the material at high loads.

As a result of an optical examination of the transverse sections through the centres of five holes of different diameters the following results (Table I.) were obtained of the differences of the principal stresses at various points in the section.

TABLE I.

Total load 100 lb. Section of specimen, 1 in. × 0·186 in.

Experiment No.	Size of hole. In.	Measured difference of the principal stresses in pounds per square inch at a distance (in inches) from the edge of—								p_m mean stress intensity, lb./sq. in.	p.
		0	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{15}{32}$		
1	$\frac{1}{16}$	545	550	...	560	...	570	615	1,470	584	549
2	$\frac{1}{8}$	560	575	...	595	...	622	1,560	...	620	547
3	$\frac{1}{4}$	580	585	...	640	760	1,770	...	...	724	568
4	$\frac{3}{8}$	625	630	...	890	1,850	...	...	...	868	570
5	$\frac{1}{2}$	635	760	1,070	2,040	...	...	...	...	1,035	613

Some of these values are plotted in Fig. 1, and a comparison of these numbers with the values obtained by calculation shows that, when the hole is small compared with the breadth of the plate, there is a very close agreement.

It may be remarked that the maximum values are difficult to obtain, as they are point values at the periphery of the hole, where the variation of stress in a radial transverse section is very great, and determinations are therefore more liable to error here than at any other point.

In order to determine the law of variation of stress along the cross-section the stresses were calculated for the ideal case of a

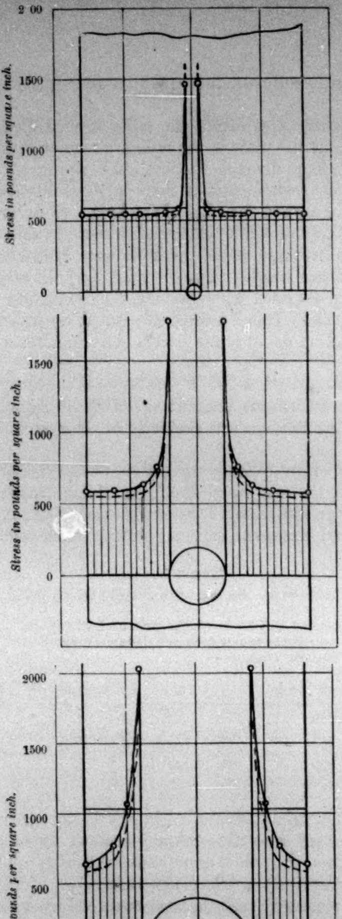

FIG. 1.—STRESS DISTRIBUTION IN A TENSION MEMBER WITH A CENTRAL HOLE. NOS. 1, 3 AND 5 OF TABLE I. (The calculated values are shown by dotted lines.)

hole in an infinite plate subjected to uniform tension, and these were compared with the values obtained in Table I. The principal stresses $\widehat{\theta\theta}$, $\widehat{rr}$ at the cross-section are given by

$$\widehat{\theta\theta}=\frac{p}{2}\left(2+\frac{a^2}{r^2}+\frac{3a^4}{r^4}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\widehat{rr}=\frac{3}{2}p\left(\frac{a^2}{r^2}-\frac{a^4}{r^4}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and from these we obtain the following ratios for values of r/a between 1 and 10.

TABLE II.

r/a …………	1·0	1·25	1·5	2·0	3·0	4·0	5·0	10·0	∞
$\widehat{\theta\theta}/p$ ……….	3	1·934	1·519	1·219	1·074	1·050	1·022	1·005	1
$\widehat{rr}/p$…………	0	0·346	0·370	0·281	0·148	0·090	0·058	0·015	0
$\widehat{\theta\theta}-\widehat{rr}/p$……	3	1·588	1·149	0·938	0·926	0·960	0·964	0·990	1

The experimental determinations differ from the ideal case in that the uniform stress p, corresponding to a plate of infinite width, is not directly ascertainable, and in order to obtain its value for any case we may select two or more experimental values and compare them with the coefficients in Table II. to obtain a value of p. If this is done it will be found that the curves of stress variation agree very well with the experimental values, but this is not a very accurate method, and a more searching test is to determine the value of p from the load applied to the plate, assuming that the principal stress variation $\widehat{\theta\theta}$ follows the law given by equation (6). If the mean stress across the section is p_m and the edge of the plate is at a distance $c\,.\,a$ from the centre we have

$$p_m c(a-1)=\int_a^{ca}\widehat{\theta\theta}\,.\,dr=\int_a^{ca}p\left(1+\tfrac{1}{2}\frac{a^2}{r^2}+\frac{3}{2}\frac{a^4}{r^4}\right)dr,$$

from which we readily obtain a formula for calculating p in terms of p_m and the dimensional value c in the form

$$p_m=p\left(1+\frac{1}{c}+\frac{1}{2c^2}+\frac{1}{2c^3}\right). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The values of p, for each case, have been determined from this latter equation, and the calculated stresses, corresponding to the observations, are plotted for comparison with the observed values, and are shown by dotted curves in each figure.

As will be seen from Fig. 1, there is a very fair agreement for small holes and a less good agreement for large ones, but these latter obviously lie outside the limits contemplated by the mathematical solution. It appears that, for any central hole having a diameter not greater than one quarter of the width of the plate or member, we may assume the law of variation of stress of equation (6) without serious error.

Within these limits the maximum stress may therefore be calculated from the applied load by the formula

$$p_{\text{max.}}=\frac{6c^3}{2c^3+2c^2+c+1}p_{\text{mean}}, \quad . \quad . \quad . \quad . \quad (9)$$

and if c is great compared with unity this reduces to the simple approximation formula

$$p_{\text{max.}}=\frac{3c}{c+1}p_{\text{mean}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the hole has a greater diameter than one quarter of the width of the plate, the experiments show that the maximum stress is greater than that calculated by these approximate rules.

Another case of importance is that of a semicircular notch in a tension member.

A perfectly general solution has been obtained by Chree,* and an approximate solution is given by Leon,† for a small semicircular notch in a tension member on the assumptions that the radial and tangential shifts with reference to the centre of the semicircle are expressed by

$$\left.\begin{aligned} u&=f_1(r)+f_2(r)\,.\,\sin^2\theta \\ v&=f_3(r)\,.\,\sin 2\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

He obtains

$$\left.\begin{aligned} \widehat{rr}&=\frac{p}{2}\left(\frac{a^2}{r^2}-\frac{a^4}{r^4}\right)+p\left(\frac{a^4}{r^4}-2\frac{a^2}{r^2}+1\right)\sin^2\theta \\ \widehat{\theta\theta}&=\frac{p}{2}\left(\frac{a^4}{r^4}+\frac{a^2}{r^2}+2\right)-p\left(\frac{a^4}{r^4}+1\right)\sin^2\theta \\ \widehat{r\theta}&=\frac{p}{2}\left(-\frac{a^4}{r^4}+\frac{a^2}{r^2}+1\right)\sin 2\theta, \end{aligned}\right\}, \quad . \quad (12)$$

which satisfy the body stress equations, but not all the surface conditions. These equations may be used for comparison with the experimental values, and they indicate that the stress is doubled at the notch, for if $r=a$, we have

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* "On the Equations of an Isotropic Elastic Solid in Polar and Cylindrical Co-ordinates," by C. Chree, M.A., "Trans." Cambs. Phil. Soc., Vol. XIV., 1889.

† "Osterreichische Wochenschrift für den Öffentlichen Baudienst," February, 1908.

$$\widehat{rr}=0, \quad \widehat{\theta\theta}=2p(1-\sin^2\theta), \quad \widehat{r\theta}=p\sin\theta \,.\, \cos\theta,$$

so that when $\theta=0$, $\widehat{\theta\theta}=2p$, and this is the maximum value obtained.

The general values of the principal stresses at the section passing through the centre of the notch and perpendicular to the direction of the stress are

$$\widehat{\theta\theta}=\frac{p}{2}\left(\frac{a^4}{r^4}+\frac{a^2}{r^2}+2\right), \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\widehat{rr}=\frac{p}{2}\left(\frac{a^2}{r^2}-\frac{a^4}{r^4}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and for purposes of comparison with the results of experiment the following table of coefficients is given :—

TABLE III.

r/a	1	1·25	1·5	2·0	3·0	4·0	5·0	10·0	∞
$\widehat{\theta\theta}/p$	2	1·525	1·321	1·156	1·062	1·033	1·021	1·005	1
$\widehat{rr}/p$	0	0·115	0·123	0·094	0·049	0·029	0·019	0·0049	0
$(\widehat{\theta\theta}-\widehat{rr})/p$ ...	2	1·410	1·198	1·062	1·013	1·004	1·002	1·0001	1

An optical examination was made of specimens having notches symmetrically arranged with respect to a transverse section as the most convenient form, and determinations of the differences of the principal stresses are shown in the accompanying Table IV., and some of these values are plotted in Fig. 2.

The solution given by equation (12) implies infinite width, but since finite width is a practical necessity it seems best on general grounds to take for experimental comparison the symmetrical form used here.

TABLE IV.

Experiment No.	Radius of notch. Ins.	Measured differences of principal stresses in pounds per square inch at a distance (inches) from the centre line of—								Mean stress intensity, lb./sq. in. p_m.	p.
		0	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{15}{32}$		
1	$\frac{1}{32}$	515	515	...	515	...	470	500	1,070	584	527
								670			
2	$\frac{1}{16}$	530	530	...	525	...	490	1,160	...	620	571
3	$\frac{1}{8}$	535	535	...	515	705	1,250	...	...	724	616
4	$\frac{3}{16}$	515	545	...	740	1,340	...	...	...	868	680
5	$\frac{1}{4}$	560	685	890	1,600	...	...	...	...	1,035	742

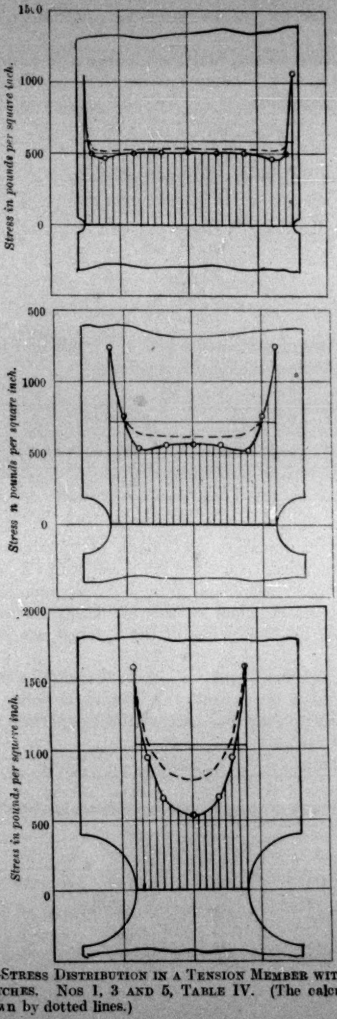

Fig. 2.—Stress Distribution in a Tension Member with Semicircular Notches. Nos 1, 3 and 5, Table IV. (The calculated values are shown by dotted lines.)

To determine how far the variation of stress agrees with the approximate expression obtained above the value of p is found from the equation

$$p_m a(c-1)=\int_a^{ca}\widehat{\theta\theta}\,.\,dr=\frac{p}{2}\int_a^{ca}\left(2+\frac{a^2}{r^2}+\frac{a^4}{r^4}\right)dr,$$

where *ca* is the half breadth of the plate from which we obtain the expression

$$p_m=p\left(1+\frac{2}{3}\frac{1}{c}+\frac{1}{6}\frac{1}{c^2}+\frac{1}{6}\cdot\frac{1}{c^3}\right). \quad . \quad . \quad . \quad (15)$$

The values of the stress variation obtained from this value of p are obtained from the last line of Table III., and are plotted for comparative purposes in Fig. 2. The agreement for small notches is fairly close, but, for those cases which clearly lie beyond the conditions assumed by the mathematical analysis, the results indicate that probably both principal stresses differ rather markedly from the calculated values. A want of agreement is also noticeable for notches of $\frac{1}{32}$ in., $\frac{1}{16}$ in. and $\frac{1}{8}$ in. radius in the neighbourhood of $r/a=2$, where the minimum values are reached, and as the radial stress rises to its maximum near this value it seems probable that the radial stress equation is not so close an approximation as the value for the tension across the section. For the larger notches the formula does not agree very well with the experimental values except at the maxima.

It is, therefore, not justifiable to state any general rule for the maximum stress due to a notch which has a radius greater than one-eighth of the breadth of the member, but within this limit we may express the maximum stress in terms of p_m and c in the form

$$p_{\text{max.}}=p_{(\text{mean})}\cdot\frac{12c^3}{6c^3+4c^2+c+1} \quad . \quad . \quad . \quad (16)$$

and if c is comparatively large so that the first power may be neglected, we have

$$p_{\text{max.}}=\frac{2c}{c+\frac{2}{3}}p_{\text{mean}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

as an approximate expression for use in calculating the greatest stress produced in a tension member with symmetrical notches of semicircular form.

A preliminary step to the solution of the second problem has been made by L. F. Richardson,* and his tentative results differ somewhat from the experimental values here found.

It is hoped that it will be possible to undertake a joint investigation by a method suggested in a former Paper † to compare with the results already obtained.

In conclusion the author wishes to thank Dr. Chree for kindly reading and criticising the manuscript.

ABSTRACT.

The necessities of practical construction lead to a number of interesting cases of stress problems in which discontinuities like holes and notches occur in great variety both as regards form and arrangement.

For the experimental determination of stresses in loaded members an optical examination of a model shaped in transparent material has many advantages.

Two cases of primary importance are examined in this way, and the results are compared with those obtained by analysis. The first example relates to the case of a hole in a tension member subjected to a uniformly applied stress, p.

The values of $(p_x - p_y)$ the difference between the principal stresses are readily obtained optically, and they show a fair agreement with the calculated values if the diameter of the hole is not greater than one-quarter of the width of the plate, but beyond this the agreement is not so good. For practical purposes it is important to be able to estimate the maximum stress from the value obtained by assuming that the total load on a tension member is uniformly distributed over the cross-section. A formula based on the relationship found in the experiments takes the form

$$p_{\text{max.}} = \frac{6c^3}{2c^3 + 2c^2 + c + 1} p_{\text{mean}},$$

where c is the ratio of the width of the member to the diameter of the hole ; if c is large compared with unity this reduces to the simple form

$$p_{\text{mean}} = \frac{3c}{c+1}.$$

In the case of two semicircular notches, arranged symmetrically with regard to the centre line and to the cross-section, there appears to be no exact mathematical solution, but an approximate one has been obtained by Leon, resulting in expressions for p_x and p_y at the minimum section of the form

$$p_x = \frac{p}{2}\left(2 + \frac{a^2}{r^2} + \frac{a^4}{r^4}\right), \quad p_y = \frac{p}{2}\left(\frac{a^2}{r^2} - \frac{a^4}{r^4}\right),$$

provided that the radius of the notch is small compared with the breadth of the plate.

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* "The Approximate Solution of Various Boundary Problems by Surface Integration combined with Freehand Graphs," by L. F. Richardson, "Proc." Physical Society, February, 1911.

† "The Approximate Arithmetical Solution by Finite Differences of Physical Problems involving Differential Equations, with an Application to the Stresses in a Masonry Dam," by L. F. Richardson, "Phil. Trans.," 1910.

Experimental determinations of $p_x - p_y$ show that the maximum values agree very well with those of the formulæ for notches having a maximum radius of about one-quarter of the breadth of the member, but the minimum values do not show a very good agreement if the notch has a radius greater than one-eighth of the breadth. The results appear to indicate that the radial stress for large notches is greater than that given by the formula.

For determining the maximum stress from the applied mean stress a formula is proposed of the form

$$p_{\max} = \frac{12c^3}{6c^3 + 4c^2 + c + 1} p_{\text{mean}},$$

and this shows a fair agreement with the experimental values.

X. *A Column Testing Machine.* *By* E. G. Coker, *M.A., D.Sc., Professor of Mechanical Engineering in the City and Guilds of London Technical College, Finsbury.*

Received November 18, 1912. Read November 22, 1912.

Experimental determinations of the strength of long members under compression loads are of considerable importance, as, in practice, members of this kind are nearly as numerous as those subjected to tension, but the results available are not nearly so complete as for tension members since compression tests are more difficult to carry out, and machines, which accurately fulfil the conditions required for experiment, are not numerous and are, moreover, very costly. It is essential in compression testing machines to ensure that the conditions of end fixing are realised, and that the load can be accurately measured. Both requirements are more difficult to carry out for compression than for tension, as a pillar under load is always tending towards an unstable condition, while a tension member acts in the reverse manner. The required conditions of end fixing can be obtained if the end plates applying the load are maintained perpendicular to the axis of the originally straight pillar, and in some small machines the end bearing plates move on slides to ensure parallelism, an obviously defective arrangement, since the friction of the slides renders it difficult to measure the applied load.

In some very large machines, for full-sized pillars, a massive head is secured by tension rods to a hydraulic cylinder, and the ram of this latter applies the load, usually in a very accurately axial direction. For rough measurements the total load is determined from the hydrostatic pressure, and is subject to a deduction for the friction of the cup leathers, usually a large correction of uncertain amount. This can, however, be very much reduced if the cup leathers are dispensed with and the ram or plunger made a very true fit in the cylinder. The plunger then simply floats in oil, which leaks slowly between it and the cylinder wall, and the friction is very small and becomes practically negligible if the ram is rotated.

In some other forms the load, applied by a hydraulic plunger of the usual type, is weighed by aid of a large circular diaphragm carrying one end plate and clamped at its edges in a casing to enclose a thin layer of liquid. The pressure registered by the imprisoned liquid enables the load to be weighed with great accuracy, while the arrangement also permits the bearing plates to remain perpendicular to the axis of the specimen. In the well-known Emery testing machine this pressure is weighed on a smaller diaphragm by a subsidiary system of levers.

A mechanical weighing device attached directly to an end-pressure plate usually necessitates this plate having more than one degree of freedom, a bad defect in a testing machine of this kind. This difficulty may, however, be overcome, and the simple machine described here uses a lever system to measure the load on a plunger, which is constrained to move axially without appreciable friction. This is accomplished by controlling its motion by aid of thin spring plates or wires, which are secured to the movable plunger and to a fixed casing, so that all degrees of freedom are suppressed except one. In one arrangement, which has been designed for testing small models of pillars made of transparent materials, the central plunger is constructed to grip the inner edges of annular discs placed with their planes a considerable distance apart and clamped to a fixed casing.

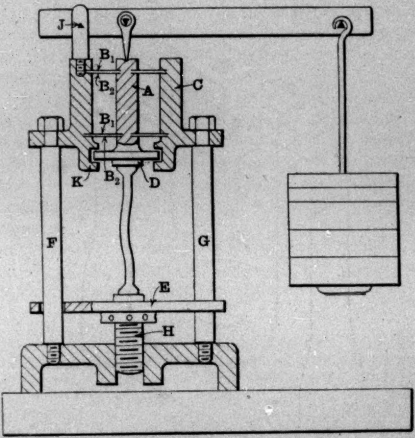

FIG. 1.

A diagram of this arrangement is shown in vertical section, Fig. 1, in which the plunger A is secured at each end to a pair of very thin brass or steel diaphragm plates, B_1, B_2, which latter are clamped at their outer edges in a casing, C, of annular cross-section. The plunger so secured has a small amount of vertical motion only, and the pillar to be tested is set between the compression plates D and E, the latter of which slides on the guide bars F and G, and is adjusted in height by a flat-topped screw, H, secured in the base.

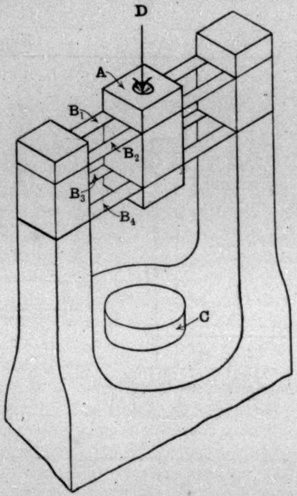

FIG. 2.

In large machines a hydraulic ram replaces the screw, and the sliding plate E is not required. The load borne by the plunger is weighed, in this example, by a single lever pivoted at J and carrying a weight at its outer end. The lever is maintained in a horizontal position by the screw, and the plunger is limited to a very small amount of motion by stops K. An index finger, not shown, is also provided to show when the plunger is freely supported by the diaphragms.

Photographs of the stress distribution show that this apparatus gives satisfactory results for transparent pillars under usual conditions of end fixture.

In some forms of apparatus it may be more convenient to use perforated diaphragms, or thin wires arranged radially in a plane. In others it may be more convenient to set thin rectangular strips in parallel planes, and the model of an arrangement for a small machine, Fig. 2, shows a block, A, supported by four steel strips B_1, B_2, B_3, B_4, arranged so that all degrees of freedom except one are suppressed. In this arrangement the load is applied by the plunger C of a hydraulic cylinder moving in the same line as the block A, and the load transmitted through the pillar to this block can be measured at the point D by a weighted lever system, or other suitable arrangement.

The application of thin plates, wires and diaphragms to give one degree of freedom to a pressure enables the load to be measured with great accuracy, since there is no appreciable loss due to friction, and the energy stored in the elastic guiding members, when these latter are displaced from their central configurations, is restored when the equilibrium positions are again attained.

In conclusion, the Author desires to express his thanks to his assistant, Mr. F. H. Withycomb, who has been very helpful in the detailed design and construction of the apparatus.

ABSTRACT.

The conditions of fixture of the ends of columns, and the large influence this has upon their strength, generally make it necessary to use special testing machines for these members, in which the end plates applying the load are accurately parallel, and remain so during a test.

If only rough measurements of the load are required this offers no serious difficulty, but accurate measurement involves elaborate mechanical devices, some of which are briefly referred to in the Paper.

This difficulty is overcome in a simple manner by supporting one pressure plate by two or more annular diaphragms spaced at considerable intervals, and clamped at their outer edges to a fixed casing in such a manner that only one degree of freedom is possible This construction is carried out in the machine described in the Paper. The total load on the pressure plate supported in this way can be measured by a loaded lever system, or other sutable means. The other pressure plate may be carried on guides, and the load applied by screws, hydraulic pressure or other suitable means.

Rectangular plates or wires may also be used in place of annular diaphragms, and a model showing an application to a compression machine is described.

Photographs of celluloid columns are shown under stress, and the colours produced by temporary double refraction indicate that the loads are satisfactorily applied in a machine of this type.

DISCUSSION.

Sir R. **Hadfield** thought this Paper opened out an important field of research.

Mr. C. E. **Larard** asked if the machine was meant for testing metals or only xylonite, as in the former case the conical pivot at the top would be a source of weakness.

The **Author** replied that for heavier machines the conical pivot could be replaced by a narrow strip of thin steel plate which he considered preferable to a knife edge.

XI. *The Electrical Conductivity and Fluidity of Strong Solutions. By* W. S. TUCKER, *B.Sc.*

RECEIVED NOVEMBER 22, 1912. READ JANUARY 24, 1913.

IT may be fairly assumed that when solutions are of such concentration that the molecules of dissolved substance occupy a relatively large volume, the ions produced will interfere with each other's movements. If the ionisation factor be calculated from the ratio of the molecular conductivity to that at infinite dilution, we get a result which fails to take into account this interference, and therefore cannot be regarded as reliable.

In Prof. Callendar's association theory of strong solutions* the depression of freezing point and elevation of boiling point can be foretold by associating with each molecule and ion a certain constant number of water molecules. With the data at his disposal he obtains good agreement between the experimental values and his calculated values for cane sugar up to any concentration, and for solutions of potassium, sodium and magnesium chlorides. For the strong solutions of calcium chloride discrepancy occurs, most probably owing to unsatisfactory ionisation data.

The experiments here described were made to find whether there is any relation between conductivity and fluidity.

Certain experiments have already been performed by Lyle and Hosking † with sodium chloride, and later by Hosking‡ with lithium chloride, but they did not work with so many strong solutions, nor did they cover the same ranges of temperature.

Calcium chloride solution, with its low cryohydrate point and its extreme solubility, forms a very good solution to work on. Moreover, it can be highly supercooled. This can be illustrated by taking a solution slightly above cryohydrate strength and cooling it.

Referring to the curves shown in Fig. 1, crystals of the salt $CaCl_2 6H_2O$ should separate out at P, but the solution supercools to Q. The addition of a fragment of ice causes rise to R, giving the approximate freezing point. Ice now separates from R to S. At S fresh crystals presumably of the salt make their appearance, causing a rise to T, probably above cryohydric point; but separation of these crystals quickly brings the solution to the constant temperature of the cryohydric point.

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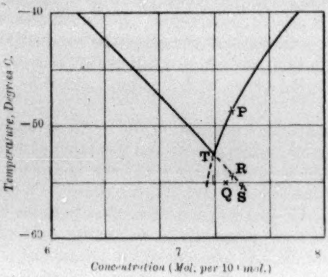
* Callendar, "Proc." Roy. Soc., A, Vol. 80, 1908, p. 466.

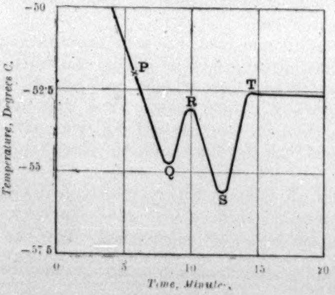
† Lyle and Hosking, "Phil. Mag.," May, 1902, p. 487.

‡ Hosking, "Phil. Mag.," May, 1904, p. 469.

A supercooling of 15°C. can be obtained with proper precautions, when the crystals are dissolved in their own water of crystallisation.



Freezing Point and Solubility Diagram.



Cooling Curve.

FIG. 1.

The feature of the method here described is the simultaneous observation of temperature, viscosity and resistance. The electrolytic cell and viscometer are immersed in the solution for which they are employed, and both are bound to the bulb of a platinum thermometer, which serves also as a stirrer.

Measurement of Conductivity.

The electrolytic cell C is a flattened tube drawn out into a flat jet. It contains a platinised-platinum electrode, P, whose dimensions are 25 mm. by 5 mm. The second electrode, also of platinised-platinum, forms a cylindrical sheath, Q, round the bulb of the thermometer T. (*See* Fig. 2.)

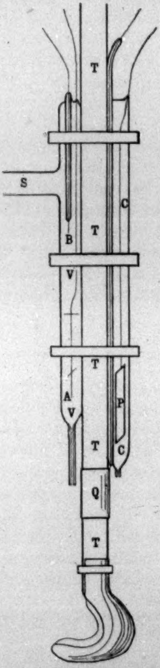


FIG. 2.—APPARATUS FOR DETERMINING VISCOSITY AND CONDUCTIVITY.

The thermometer is mounted on a frame to which a vertical oscillatory motion is imparted. This motion is caused by a connecting rod attached to a large pulley driven by a motor.

As the thermometer oscillates it draws through the liquid a platinum scoop, which acts as an efficient stirring arrangement.

The resistance was measured by that modification of Kohlrausch's method described by Fitzpatrick,* in which a high-speed double commutator and a moving-coil galvanometer replace the coil and telephone. An accuracy of 1 part in 2,000 could be thus obtained, and was greater than that possible with the viscosity apparatus.

A preliminary experiment was made which showed that the oscillatory motion of the cell and the consequent motion of the liquid in it did not affect the resistance readings. Such oscillatory motion ensured the temperature of the liquid in the jet being exactly that of the liquid in the bath.

In the observations taken the concentration was expressed in gram-molecules per 100 molecules of water. This concentration remains constant with change of temperature, and was employed by Prof. Callendar in his Paper on strong solutions.

The capacity of the cell was found by the use of a standard solution employing Kohlrausch's results.

The molecular weight of the anhydrous salt is taken as $110{\cdot}9_2$.

Measurement of Viscosity.

The viscosity is measured by a modification of Poiseuille's method. The tube through which the flow occurs is a uniform jet about 2 cm. long drawn out from a piece of quill tubing, V, as shown by the diagram, Fig. 2. Connection can be made by means of the side-tube S with an aspirator or with a compressed air reservoir. When the tube is immersed in the solution, the latter can be drawn in or driven out at will.

The viscometer is plunged into the solution to a certain depth, and the time of inflow is measured and compared with that of water under the same conditions.

During the experiments the viscometer is invisible, hence the levels of the liquid when flowing into the tube are indicated by electrical means.

Two fine platinum wires A and B serve as contacts and are insulated from one another. As the liquid rises within the tube it makes circuit with Q and A through the liquid and the motion of a galvanometer is noted. A second motion is produced when contact with B is made. Thus the passing of the levels can be accurately timed.

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* B.A. "Report," 1886, p. 328.

The assumption is now made that the rate of influx of the solution is proportional to the hydrostatic pressure, and inversely proportional to the viscosity.

The accuracy of this assumption was checked by an experiment with pure water, whose viscosity has been accurately found by Thorpe and Rodger.*

Two experiments are performed.

1. The water is run into the tube under its own pressure, the initial and final differences in level being H and h_1 respectively. It can be shown that

$$\frac{H}{h_1}=e^{\frac{K}{a}g.\frac{\rho}{\eta}t_1}=e^{mt_1},$$

where ρ is the density, η the viscosity, t the time of inflow, K a constant depending on the dimensions of the jet, and a the area of the wider portion of the tube.

2. An external pressure P is applied by means of the aspirator, and this acting with the hydrostatic pressure accelerates the rate of inflow. The expression then becomes

$$\frac{\frac{P}{\rho}+H}{\frac{P}{\rho}+h_1}=e^{mt_2},$$

where t_2 is time of inflow.

Readjustments are made so that $h_1=0$, and H becomes H_1,

giving $$1+\frac{\rho H_1}{P}=e^{mt_2},$$

from which $$\rho=\frac{P}{H_1}\{e^{mt_2}-1\};$$

but $$m=\frac{1}{t_1}\log_e\frac{H}{h_1},$$

hence $$\rho=\frac{P}{H_1}\left\{e^{\frac{1}{t_1}\log_e\frac{H}{h_1}t_2}-1\right\}.$$

With the apparatus used a value of the density of water was obtained with an error of less than 1 per cent. This result justifies the use of the method, since no higher accuracy was attempted. The viscosities of the solutions used in the extreme cases are as 1 : 40.

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* Phil. "Trans.," 1894, A, p. 1.

Water is taken as a standard liquid of known viscosity.

Variation of Conductivity and Fluidity with Concentration at Constant Temperature.

The following table shows the results obtained for solutions of different concentrations all at the temperature of 16·77°C.

This temperature was approximately that of the room. To ensure its constancy the solution was contained in a cylindrical Dewar vessel, which, however, was not silvered. The temperature of the contents was brought to the above exact value by presenting to the walls of the vessel a warm or cool surface, and the liquid was stirred to secure uniformity of temperature. The vessel was then screened from further radiation.

TABLE I.

Temperature 16·77°C.

(1) Concentration. (n) Mol. per 100 mol. of water.	(2) Density. Grammes per cubic cm.	(3) Resistance. Ohms.	(4) Conductance. (C). Ohms^{-1} $\times 10^{-4}$.	(5) Viscosity. C.g.s. units.	(6) Fluidity. (F) C.g.s. units.	(7 $\frac{1}{n} \cdot \frac{C}{F}$ $\times 10^{-}$
0	0·9988	...	...	0·01081	92·50	...
0·927	1·0434	2,032	4·921	0·0121	82·64	6·424
2·757	1·1293	915·4	10·93	0·0161	62·11	6·38
4·106	1·1793	782·5	12·78	0·0199	50·25	6·19
5·436	1·2313	766·8	13·04	0·0251	39·84	6·02
7·252	1·2933	845	11·83	0·0371	26·95	6·05
8·429	1·3308	968·5	10·32	0·0486	20·58	5·95
9·363	1·3569	1,074	9·31	0·0602	16·61	5·99
10·11	1·3788	1,179	8·479	0·0706	14·16	5·93
10·65	1·3960	1,278	7·823	0·0818	12·22	6·01
11·22	1·4063	1,435	6·969	0·0989	10·11	6·14
*16·66	1·5210	3,220	3·106	0·3320	3·012	6·18

* Melted crystals supercooled about 13°C.

The meaning of the last column is difficult to interpret, but is worth quoting. It will be seen that while the concentration alters from 4 to 16 molecules, while fluidity changes to 16 times its value and conductivity suffers a fourfold change, the quantity $\frac{1}{n} \cdot \frac{C}{F}$ only varies about 3 per cent.

A similar treatment of Lyle and Hosking's results for sodium chloride and Hosking's results for lithium chloride suggests a similar conclusion. The data supplied, however, are inadequate, since the solutions are generally more dilute.

Curves connecting C and n, F and n, and $\frac{C}{F}$ and n are plotted in Fig. 3.

From the fluidity and conductivity curves no evidence is given of a vanishing of these quantities at higher concentrations.

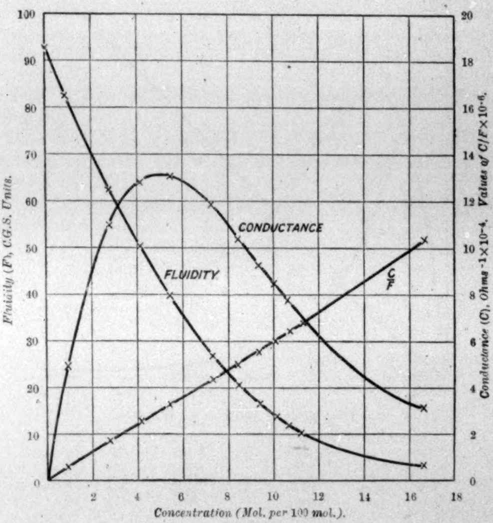

FIG. 3.

The above authors quoted such a vanishing of both quantities for sodium chloride at 10·74 normal, and for lithium chloride at 16-normal concentration.

Both these results are obtained by extrapolation.

[Since this Paper was first submitted, the results obtained by Bousfield and Lowry* with strong solutions of sodium hydrate have been examined. The relations between conductivity, fluidity and concentration expressed as a ratio of weights of solid to solvent are shown in Fig. 3A.

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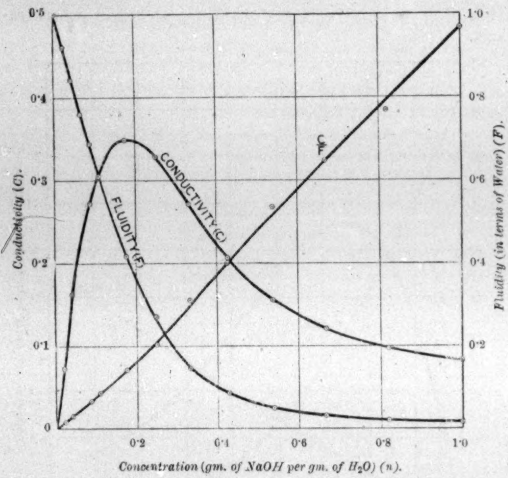
* Bousfield and Lowry, "Phil. Trans.," 1905, Vol. 204, p. 253.

The linear relation between $\frac{C}{F}$ and n for strong solutions is again shown.]



n	$\frac{C}{nF}$	n	$\frac{C}{nF}$
0·111	4·57	0·538	5·98
0·25	4·86	0·666	5·84
0·333	5·63	0·818	5·97
0·428	5·60	1·000	5·79

FIG. 3A.—PROPERTIES OF NaOH SOLUTIONS (BOUSFIELD AND LOWRY).

Variation of Fluidity and Conductivity with Temperature.

The same apparatus was used as before, except that the electrode P in the electrolytic cell was slightly moved.

The temperatures above that of the air were obtained by surrounding the Dewar cylinder by a second larger Dewar cylinder containing water.

This was heated by a current of steam.

For temperatures below that of the air, ice was added to the water, and below 0°C. a mixture of ice and salt was employed. Finally, for temperatures below −10°C. liquid air was used. After cooling to the desired temperature the excess of liquid air was boiled off by passing a current of air through it. During the observations the mean of the initial and final temperatures gave the temperature quoted.

In the viscosity-temperature experiment the solution of strength 7·252 molecules was taken. A preliminary experiment was made to determine the densities at the various temperatures employed. For this purpose an accurate mercury thermometer, reading to 50°C. by $\frac{1}{10}$ deg. divisions, was employed. The mercury was replaced by the solution and the volume of the bulb was accurately found by weighing the mercury.

The co-efficient of expansion for a range of nearly 100°C. is nearly constant, its mean value between 0°C. and 50°C. being 0·000455.

The following table gives the relation between fluidity and temperature :—

TABLE II.

Concentration 7·252 molecules per 100 molecules of water.

(1) Temperature. °C.	(2) Density.	(3) Viscosity.	(4) Fluidity. C.g.s. units.
85·22	1·259	0·01438	69·55
70·61	1·266	0·01678	59·59
56·79	1·275	0·01849	54·09
37·15	1·282	0·02666	37·51
23·70	1·290	0·03544	28·22
16·77	1·293	0·0418	23·92
−0·15	1·301	0·0726	13·77
−5·51	1·303	0·086	11·56
−8·85	1·305	0·1007	9·94
−15·19	1·308	0·1338	7·475
−19·07	1·309	0·1569	6·372
−30·45	1·314	0·2628	3·806
−49·16	1·323	0·5919	1·689

At the same time, readings of resistance were taken from which the conductivity for the same solution was obtained. These are included in Table III. (p. 120), together with results for the other solutions.

Columns (1) and (4) of Table II. and (1) and (2) of Table III. are plotted in Fig. 4, while columns (1) and (3) of Table III. are plotted in Fig. 5.

TABLE III.

(1) Temperature. °C.	(2) Specific conductivity of solution 7·252 mol.	(3) Values of $\frac{\text{conductivity}}{\text{concentration}}$ at the concentrations given.								
		0·927	2·13	5·436	7·252	8·22	9·363	10·11	10·65	11·22
39·72	2,280	1,034	922	455	314·0	250·1	198·0	166·7	150·3	132·4
31·74	2,000	909	808	398	276·0	217·0	170·2	144·0	143·4	112·2
27·67	1,858	853	742	371	256·0	200·0	157·6	130·5	119·2	103·4
23·69	...	...	...	...	...	...	...	...	108·0	92·0
19·74	1,598	739	628	320	220·0	170·0	143·6	110·0	...	85·1
15·78	1,457	677	558	295	201·0	153·3	121·0	100·0	88·3	77·7
11·70	1,340	617	...	268	185·0	142·1	106·6	89·5	...	69·0
9·00	...	...	492	...	...	...	...	...	...	...
5·79	...	522	...	...	...	...	...	...	...	57·1
4·09	...	...	...	...	...	111·5	86·0	...	...	...
3·81	...	505	...	222	...	...	...	...	...	...
—0·14	997	455	400	203	137·5	103·0	77·7	62·6	54·6	...
—4·37	880	...	...	180	121·4	89·6	68·3	54·6	...	...
—8·28	774	...	340	159·5	106·6	78·9	...	...	...	...
—10·24	...	...	316	...	...	...	554·0	...	...	...
—12·20	683	...	...	142	94·2	69·2	...	...	...	...
—16·14	600	...	...	124·2	82·7	59·4	...	...	...	...
—20·17	512	...	...	107·8	70·7	50·8	...	...	...	...
—24·11	435	...	...	97	60·0	43·0	...	...	...	...
—27·99	369	...	...	79·3	50·9	35·7	...	...	...	...
—31·88	307	...	...	67·1	42·5	...	...	...	...	...
—33·94	...	...	...	...	...	26·3	...	...	...	...
—35·84	246	...	...	...	34·0	...	...	...	...	...
—39·78	199	...	...	...	27·5	...	...	...	...	...
—43·65	149	...	...	...	20·5	...	...	...	...	...
—47·55	114	...	...	...	15·7	...	...	...	...	...
—51·5	81·9	...	...	...	11·0	...	...	...	...	...

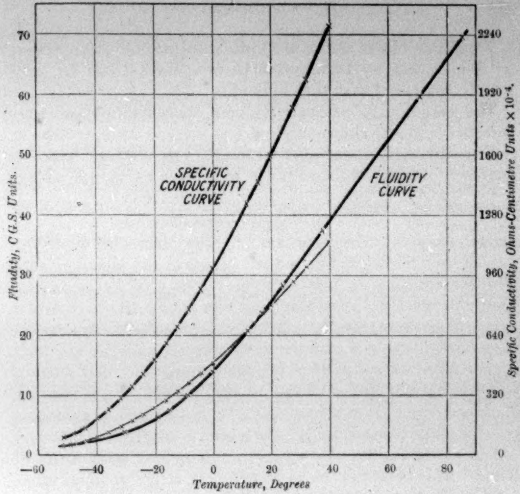

SPECIFIC CONDUCTIVITY CURVE
FLUIDITY CURVE
Fluidity, C.G.S. Units.
Specific Conductivity, Ohms-Centimetre Units × 10⁻⁴.
Temperature, Degrees
0
10
20
30
40
50
60
70
0
320
640
960
1280
1600
1920
2240
−60
−40
−20
0
20
40
60
80

FIG. 4.

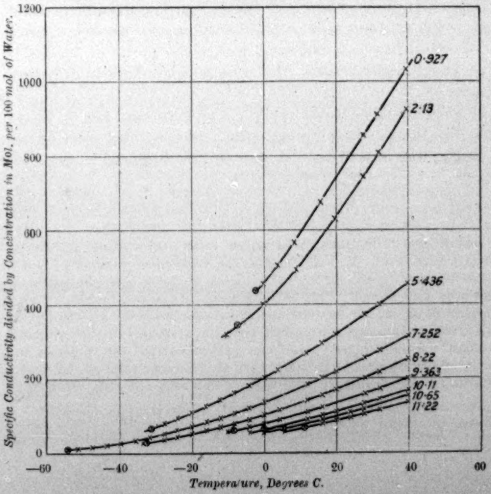

Specific Conductivity divided by Concentration in Mol. per 100 mol. of Water.
Temperature, Degrees C.
0·927
2·13
5·436
7·252
8·22
9·363
10·11
10·65
11·22
0
200
400
600
800
1000
1200
−60
−40
−20
0
20
40
60

FIG. 5.

Fig. 4 has a third curve, the ordinates for conductivity being halved, so that its variation with temperature can be more readily compared with that of fluidity.

The variation of conductivity with concentration and temperature is clearly shown in Fig. 5.

Very dilute solutions give a straight line relation, which is gradually modified to a curve of increasing curvature, as the concentration increases. The ratio $\frac{\text{conductivity}}{\text{concentration}}$ has here no definite meaning, since the denominator deals with a ratio of masses, but the latter has the advantage of not changing with temperature, and in this way the curves can be separated. They are very similar to the molecular conductivity curves.

It will be seen that there is no evidence of a disappearance of conductivity at a definite temperature, such as has been suggested by Lyle and Hosking for sodium chloride solutions and by Hosking for solutions of lithium chloride. The risk in extrapolation from above −20°C. to obtain such a vanishing point is clearly shown by the lower curves of Fig 5.

The above results also agree in this matter with those of Kunz,* who worked with sulphuric acid at low temperatures. As distinct from sulphuric acid, however, these properties of calcium chloride solutions exhibit no marked breaks in continuity with concentration at the temperatures above referred to. We thus have no evidence, within the limits of accuracy of the above observations, of the existence of many different hydrates.

The author wishes to thank Prof. Callendar and Dr. S. W. J. Smith for their valuable suggestions and helpful interest in the work, which was carried out at the Imperial College of Science.

ABSTRACT.

In adopting Callendar's association theory of strong solutions ("Proc." Roy. Soc., A., Vol. LXXX., p. 466) some difficulty is experienced in getting the strongest solutions of electrolytes to conform to the laws laid down. This is attributed to the inaccuracy of the ionisation data, which are derived from observations on electrical conductivity. It may be supposed that the viscosity of the solution will affect its conductivity, and the author carried out a series of experiments to determine if there were any definite relation between conductivity and fluidity in the case of calcium chloride solutions.

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* Kunz, "Compt. Rend.," Vol. CXXXV., p. 788, 1902.

The feature of these determinations is the simultaneous observation of viscosity, electrolytic resistance and temperature.

Solutions were contained in an unsilvered Dewar cylinder, which permitted a slight adjustment of temperature if necessary.

A platinum thermometer records the temperature, and is caused to oscillate in the solution by a stirring mechanism.

In so doing it draws a platinum scoop through the liquid and thus acts as an efficient stirrer. The conductivity cell and the viscometer were bound to the thermometer by rubber bands. While the thermometer oscillates the readings of electrical resistance were measured by means of the rotating commutator and bridge. The viscometer was in the form of a capillary pipette, which was immersed in the solution a known depth. The levels of the inflowing liquid were indicated electrically, and hence the rates of inflow could be accurately estimated and compared with that of water. Viscosities correct to less than 1 per cent. were obtained

Isothermal Observations.—Perfectly smooth curves for conductivity and fluidity were obtained, even when the supercooled melted crystals were included. No definite connection between conductivity, fluidity and concentration can be derived if the latter is expressed in terms of volume, but if concentration is expressed as a ratio of masses—molecules of solute to 100 molecules of solvent—the ratio conductivity C /fluidity F stands in linear relation to the concentration n when the latter exceeds one-fourth its maximum value. In spite of the enormous variations of the quantities in this ratio C/nF has values differing at most only 2 per cent. from the mean.

Examination of similar results by Bousfield and Lowry for sodium hydrate give striking agreement with this conclusion. However, owing to the dependence of C and F on linear dimensions, this relation is difficult to interpret.

Variations with Temperature.—One solution of nearly cryohydric strength was examined at temperatures from 40°C. to −50°C. The same tube and apparatus were employed, and for lower temperatures liquid air was used as a cooling agent. The failure of the fluidity-temperature and conductivity-temperature curves to exhibit the same variations was clearly shown.

Conductivities of various solutions were examined from 40°C. to their freezing points and the curves C/n and temperature plotted. The increasing curvature with concentration is shown and the error involved in applying the ratio, molecular conductivity to that at infinite dilution, obtained at one temperature, to indicate ionisation at another temperature, is quite apparent.

Moreover, within the limits of accuracy employed no indication is given of the existence of different hydrates.

The results obtained above suggest that no reliance can be placed on ionisation data derived from electrical conductivity observations.

DISCUSSION.

Prof. C. H. Lees remarked that the observations should be extended to see whether other strong solutions gave a simple relation between the fluidity and the resistance.

Mr. F. P. **Worley** thought the ratio of the conductivity to the concentration when the latter was expressed as so many gramme molecules of salt per 100 of water had a rather doubtful meaning.

Mr. F. E. **Smith** asked if Mr. Tucker had got any explanation of the fact that he found the cryohydric point 2½ deg. higher than previously found.

The **Author**, in reply, stated that Morse and Fraser, working on the osmotic pressure of sugar solutions, found it bore a simple relation to the concentration per 100 molecules of water and not to the volume concentration. In reply to Mr. F. E. Smith, Roozeboom had used an alcohol thermometer in determining the cryohydric point. He might also have got it too low, due to supercooling.

XII. *Some Methods of Magnifying Feeble Signalling Currents.**

By S. G. Brown.

Telegraphy over long submarine cables is continually on the increase, and I think it may be brought forward as a fairly accurate statement that the number of messages sent doubles itself every ten years. It is therefore important that, besides the increase in the number of the cables laid down each year, means should be devised to increase the carrying power.

The instruments which I have invented and am about to describe were designed primarily for cable work, but they are equally applicable to recording many other kinds of signalling impulses.

For good reasons, recording by photographic means is objected to by nearly every telegraphist. If the photographic method were permissible, great advances in speed would be available, but it is important that the record should be of a simple, cheap and immediate nature.

Lord Kelvin invented the siphon recorder in 1867—that is, about 45 years ago; he designed it so carefully that no improvement in its sensitiveness has been brought about until now.

Short Siphon Recorder.

In siphon recorders of the moving coil type what has to be done consists of—

1. Overcoming the inertia of the coil and siphon.
2. ,, ,, back E.M.F. of the coil.
3. ,, ,, control of the suspensions.
4. ,, ,, friction of air, suspensions and inking.

As the siphon has to return to zero in a certain time after the current in the coil ceases, it is necessary for the coil and siphon to have a definite frequency of oscillation depending on the speed of the signals. For submarine telegraphy this frequency lies between about 3 and 10 per second, and is adjusted by varying the control on the coil. As the control necessary to give a certain natural period to the moving system is proportional to its moment of inertia, it follows that by reducing this inertia we reduce the forces required both to accelerate the coil and to overcome the control.

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* Discourse delivered at the Eighth Annual Exhibition of Apparatus, held by the Society on December 17, 1912.

The ordinary siphon recorder employed is a siphon tube of about $2\frac{1}{2}$ in. long and from 8 to 12 mils in diameter. The moving coil consists of 500 turns of 2-mil wire at a mean radius of $\frac{3}{8}$ in. The coil and siphon are mounted on separate axes and are connected by silk fibres so that the angular movement of the siphon is about two to three times that of the coil. By reducing the length of the siphon to $\frac{1}{2}$ in. and substituting a narrower coil it is possible to greatly increase the sensitiveness of the recorder.

In order to make the inertia effects of the moving system a minimum, it is advisable to make them equal for the coil and the siphon. Even a narrow coil of 300 turns has about 100 times more inertia than the siphon, so that it is necessary to move the siphon through $\sqrt{100}$ times the angle moved by the coil.

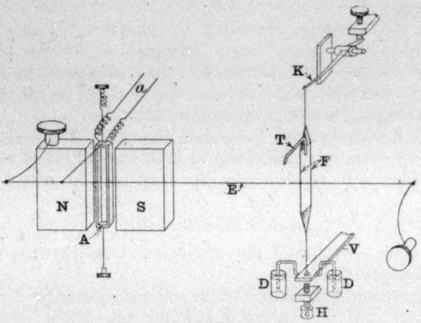

FIG. 1.

By reducing the number of turns on the coil and increasing the field it is possible to reduce the natural period for a given sensitiveness and back E.M.F., but as the mass of the mountings and insulation of the coil only decrease slightly as the turns are reduced the gain is not very marked. In practice it is inadvisable to reduce the turns on the coil below 50 or 100 turns, as with lower values the power required to overcome the friction of the air and inking becomes too limited. This precludes the possibility of attaching the siphon directly to a coil of a few turns, and means of magnifying the motion of the coil and transmitting it to the siphon have to be used. In this instrument (Fig. 1) it is accomplished by means of a fine fibre, E, which is kept in tension by flat springs at each end. The fibre is attached to an arm carried by the moving coil A, and to a vertical fibre, F, on the siphon suspension.

The siphon is carried on an aluminium carrier to which a single central fibre is attached at the top and two parallel fibres, FF, 0·2 in. apart below. One leg of the siphon (Fig. 2) lies on the axis of the suspension and dips into a small opening in a pipe extending from the ink-pot. This arm goes in between the two vertical fibres, and as the opening in which the siphon dips is only a small one, the ink level remains practically constant whether the reservoir is full or not. The siphon turns round on the axis in which the leg lies, and this makes the drag between the moving siphon and the ink very much less than if the siphon cut across the surface of the ink.

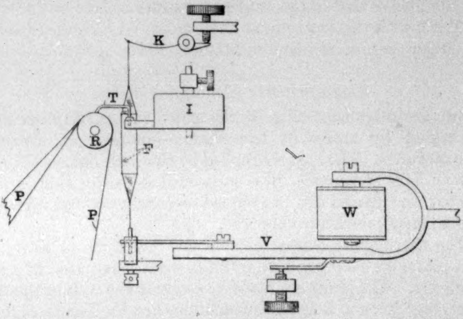

FIG. 2.

In order to produce an ink line on the paper without introducing friction the siphon must not touch the paper even momentarily, and arrangements have been made to jerk the ink in fine drops on to the paper. To accomplish this the whole of the siphon suspension is vibrated rapidly up and down between the springs V and K by means of the spring V, which is attached to the vibrator. As the spring V is very weak in comparison with the reed, the vibrations of the latter are not affected by the movements of the spring. To impart a jerk to the siphon a stop, H, is fixed directly under the axis of suspensions and two little dash-pots, DD, on either side prevent the spring bouncing on the stop.

The working end of the siphon is ground flat, and an aniline dye with a small proportion of methylated spirit or ordinary red ink is used for recording on the paper. In this way a fine line of very closely spaced dots can be obtained without introducing any appreciable drag on the siphon.

For signalling purposes, the distortion due to the radius of the siphon being only ½ in. is not at all troublesome as the velocity of the paper moving round the wheel R masks this.

When the instrument is adjusted to have a natural frequency of 10·5 per second, with a 300-ohm 300-turn coil, a current of 50 microamperes gives a full-sized signal corresponding to a deflection of 0·1 in. on the paper. Under these conditions the back E.M.F. of the coil is only about one-quarter to one-fifth of that of the ordinary recorder coil.

Trials with this instrument have shown an increase of speed of 30 per cent. on the largest Atlantic cables.

Thermo-electric Magnifying Relay.

In this instrument (Fig. 3) the power in the relay circuit is generated by means of five thermo-junctions at different temperatures. The heat is supplied by two little flames, CC, and a very light thermopile, B, is suspended so as to swing in and out of the flames and is coupled to a moving coil through which the received currents pass.

The thermopiles consist of alternate junctions of platinum and platinum + 20 per cent. iridium, wires being used of 1 mil diameter. The joints are made by twisting the ends of the two wires together and holding the junctions in a Bunsen-flame for a short time. In this way a perfectly good and permanent joint is ensured. The wires are melted on to a fine glass tube about 10 mils in diameter, and one connection is brought down inside the tube to the first junction and the other connection comes along the outside of the tube.

For moving the thermopile in the flames similar arrangements to those just described for the siphon recorder are employed. Under the saddle which carries the thermopile the two silk fibres are stretched, and on to one of these the cross fibre which transmits the movements of the coil to the thermopile is attached. The top and bottom suspensions are of fine phosphor bronze wire and serve as leading-in wires to the thermopile.

To supply the heat two little flames are fed by two or three strands of cotton wick with alcohol or methylated spirit. If the wick just protrudes above the opening a small steady flame is produced and the lamp is provided with adjustments to vary the distance between the flames and the position of both flames relative to the thermopiles.

Instead of burning directly on the lamp wicks, a simple vapour burner can be fitted which will give good results even with very impure spirit. This consists of a brass cap which is kept hot by a copper wire attached to it at one end, and is heated at the other end by the flame. By altering the amount of wire in the flame the size can be varied.

An alternative arrangement which gives greater sensitiveness and enables heavier thermopiles to be used is to fix the thermopiles and vary the flames by means of a valve or shutter actuated by the coil movements.

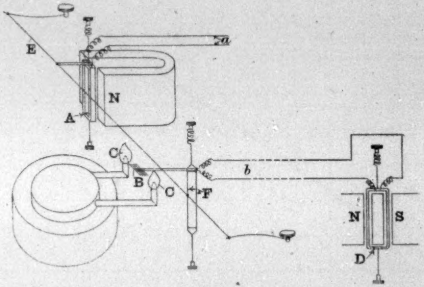

Fig. 3.

As the thermopile current depends on the difference of temperature between the junctions a certain time is required to heat the wires. It is found that for cable work, where the frequency seldom exceeds 10 per second, the lag is inappreciable, but for considerably faster movements it becomes important.

In duplex working when the sending current has to be balanced so as not to affect the receiver, quick "jarry" movements are very difficult to eliminate, but the lag in the thermo instrument reduces these movements very considerably and is a valuable property.

When the thermopile is in its central position and no current is flowing both junctions are at a dull red heat, and when fully deflected one junction becomes bright red and the opposite one is black or very faintly red. In intermediate positions the current generated by the thermopile is nearly proportional to the deflection.

The curve (Fig. 4) was taken from a thermopile with seven junctions on each side. When the thermopile was deflected 0·075″ the current it sent through a resistance of 42 ohms (equal to its own resistance) was 0·81 milliampere. With the natural period of the coil equal to 8·7 per second and a 480-ohm 480-turn coil, a current of 0·03 milliampere through the coil gave a current of 0·81 milliampere from the pile through an external resistance of 42 ohms. For slowly changing currents this corresponds to a magnification of power of about 27 times, and of course this can be greatly increased by reducing the period of the coil. For quickly changing movements the power magnification is not so great, owing to the back E.M.F. of the coil.

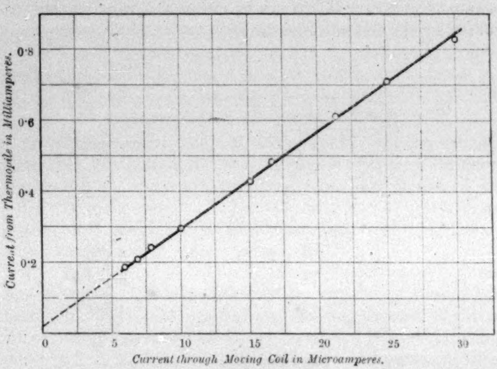

FIG. 4.

Trials of this instrument on an Atlantic cable have shown an increase in speed of about 40 per cent.

Mechanical Relay.

The instrument just described is a magnifying relay—that is to say, it multiplies the impulses received in exact proportion to their strength. This form of relay is quite distinct from an ordinary make-and-break relay, which delivers a constant current for any impulse over a certain strength. For very many purposes it is essential that received impulses should be magnified without altering their shape, and this can only be done by an instrument with a constant magnifying power.

That this is the case in the thermo relay is shown by the diagram (Fig. 4) where the current supplied to the coil and the current delivered by the thermo-junctions are plotted. Within the range of the instrument the points lie on a straight line and represent, in this case, a constant magnification in current of about 27 times.

This property I will now illustrate in an entirely mechanical relay in which movements operated by very small forces are largely increased in strength without affecting their motion. The relay consists in principle of a rotating spindle around which are wound one or more turns of a flexible cord. The spindle is revolving in such a direction as to pull away from the magnified forces and towards the small forces that control the movement. Suppose a heavy weight has to be raised by a force of one-tenth of the amount, it will obviously be necessary to supply 90 per cent. additional energy, and this is supplied by the motor driving the spindle. The magnification of force and energy depends on the number of turns which the cord makes round the spindle and follows a compound interest law.

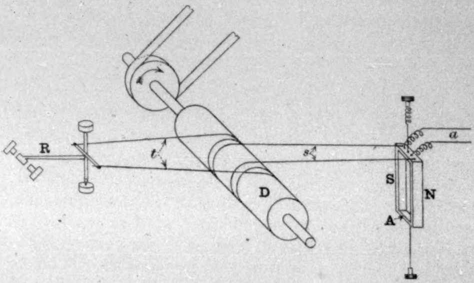

FIG. 5.

In the model shown it will be seen that a large magnification of power can be easily obtained by very simple means. Thus

I can move this 14 lb. weight rapidly up and down by pulling upon this silk fibre.

Fig. 5 shows an application of the principle to cable work, in which the small forces operating the coil A are intensified sufficiently to work the coarse relay arm R. The spindle rotates away from the relay arm R and towards the coil, and produces a much greater tension in the fibres *t* than in *s*. When the coil swings on its axis the tension is increased in one of the fibres and diminished in the other and a similar change in a magnified degree takes place in the fibres *t*.

By using means of this sort it is possible to work an ordinary siphon direct writer which normally requires some 3 milliamperes by a current of 10 microamperes

XIII. *The Resistance of Electrolytes.* *By* S. W. J. Smith, *M.A., D.Sc., and* H. Moss, *B.Sc., Imperial College of Science.*

Received January 24, 1913. Read January 24, 1913.

I. *Introductory.*

In 1911 some interesting experiments were exhibited before this Society from which the conclusion was drawn that the resistance of an electrolyte subjected to alternating currents varies with the frequency of these currents to an easily perceptible degree. A bridge method was used, a variable self-induction being inserted in the arm containing the electrolyte and a vibration galvanometer being employed as detector.*

It was assumed that the E.M.F. of polarisation could be written in the form

$$e=\frac{1}{C}\int idt,$$

in which i is the current and C the so-called capacity of the electrodes.

Now this expression takes no account of the fact that few, if any, electrolytic condensers retain their charges unimpaired when the polarising current is removed. There is in general a leakage which increases with the degree of polarisation of the electrodes. It is easy to show that the effect of such a leakage is to alter the phase of e with respect to i, so that they are not in quadrature as the above expression supposes.

In the simplest case, when $i=i_0 \sin pt$, the value of e is not

$$\frac{-i_0}{Cp}\cos pt,$$

which it would be if there were no leakage, but

$$\frac{-i_0}{C'p}\cos(pt+\varphi),$$

where C′ is obviously greater than C because the maximum value of e is less than it would be if leakage were absent.

The effect of this change in the expression for e is that (when the current $i=i_0 \sin pt$ is passing) the E.M.F. between the ends of the arm (of total resistance R) containing the electrolyte is not

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* H. F. Haworth, "Phys. Soc. Bulletin," March, 1911.

$$E = i_0\left\{R \sin pt - \left(\frac{1}{Cp} - Lp\right)\cos pt\right\}$$

but

$$= i_0\left\{\left(R + \frac{\sin\varphi}{C'p}\right)\sin pt - \left(\frac{\cos\varphi}{C'p} - Lp\right)\cos pt\right\}.$$

The condition that E and i should be in the same phase—and therefore that their ratio should be the same at any instant—is thus not

$$\frac{1}{Cp} - Lp = 0$$

but

$$\frac{\cos\varphi}{C'p} - Lp = 0.$$

Hence, when the bridge is balanced, the apparent resistance of the electrolyte arm is not R but

$$R\left\{1 + \frac{\sin\varphi}{C'pR}\right\}.$$

It will therefore appear to be a function of p if the quantity $\sin\varphi/C'pR$ is not negligible in comparison with unity.[*]

The interactions between polarised electrodes and the surrounding solutions are frequently complex and the phenomenon just indicated is only one out of several possible disturbing influences. It is difficult in any particular case to decide that all of them have been allowed for.

The cause of the leakage assumed above varies from one case to another; but it can be followed very clearly in one particular case which we shall attempt to summarise.

Consider a cell in which the electrolyte contains a salt of the metal of which the electrodes are composed. Let the concentration of the salt be c and let a small E.M.F. be applied to the cell. The current which ensues will be accompanied by changes of concentration in the border layers of the electrolyte. Let these changes be $+dc$ and $-dc$ respectively at the anode and cathode surfaces. Then, assuming the logarithmic formula due to Nernst, the expression $e \propto dc/c$ will represent, sufficiently closely, how the back E.M.F. of polarisation depends upon dc. Using alternating currents and supposing diffusion to be negligible, the concentration changes, and the back E.M.F., would acquire maximum values at the end of each half-period of the current, as the expression $e \propto \int i\,dt$ requires. But it is obvious that diffusion cannot be ignored. Its effect will be to make the concentration changes reach maximum values before the end of each half-period—namely, when the rates of gain and loss of the cathode and anode layers respectively, owing to diffusion, equal the rates of loss and gain due to the current. If we confine the concentration changes to one electrode by making the other electrode very much larger we get for the stationary state, as Warburg has shown,*

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[*] For estimates of the magnitudes which $\sin\varphi/C'$ can attain, *see* M. Wien, "Ann. der Physik," LVIII., p. 37, 1896; *also* F. Kohlrausch, *ibid.*, p. 514.

$$dc \propto \frac{-i_0}{\sqrt{p}} \cos\left(pt+\frac{\pi}{4}\right),$$

where, as before, dc is the change of concentration in the immediate neighbourhood of the electrode. If we suppose that, by the addition of a neutral salt containing the same anion, any P.D. within the electrolyte is obliterated, we get that

$$e \propto dc/c \propto \frac{-i_0}{\frac{c}{\sqrt{p}}\,p} \cos\left(pt+\frac{\pi}{4}\right).$$

Comparing this with the expression $\frac{-i_0}{C'p}\cos(pt+\varphi)$ for e given on p. 133, we see that we ought to have $\varphi=\pi/4$, that the apparent capacity $K=1/Lp^2=C'/\cos\varphi$ should be proportional to $c/\sqrt{p}$, and that the variable part of the apparent resistance, *i.e.*, $\sin\varphi/C'p=Lp\tan\varphi$, should be Lp.

These conclusions can be verified experimentally, as has been shown by Neumann.†

Next, suppose that c is reduced until it becomes almost negligible. The changes of concentration dc accompanying a given small polarisation e will now almost vanish. Diffusion effects will be reduced to a minimum, since the concentration gradients will have almost disappeared, and the apparent capacity due to concentration changes will be negligible. The polarisation (variations in the charge) of the condenser-like double layer at the electrode surface will, however, take place as before. The capacity of this layer (neglected above) is independent of c, and occasions a back E.M.F. of the simple form first mentioned. It is, therefore, to be expected that, as c gets smaller, ϕ will approach zero and the capacity a relatively small constant value independent of p. The effects produced by gradually reducing c have been studied experimentally by Krüger,* and shown to be explicable in the way here indicated.

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* E. Warburg, "Ann. der Physik," LXVII., p. 493, 1899.

† E. Neumann, "Ann. der Physik," LXVII., p. 499, 1899.

We have thought it necessary, however imperfectly, to draw attention to this work, because it seems to have attracted much less notice than it deserves. It shows the need for care in the interpretation of any new experiments upon the resistance of electrolytes.

II. *Experimental.*

It occurred to us that there was a simple way in which we could use modern instruments to find, without risk of serious error, whether the resistance of an electrolyte depends appreciably upon the frequency of the currents to which it is subjected. It depends upon direct readings of current and voltage and, consequently, is not susceptible of the highest accuracy. We thought, however, that we could expect to attain an accuracy of 1 part in 1,000 in comparing resistances, at any frequencies within the range at our disposal, by its use.

The principle of the method will be obvious from the figure (representing a particular case) given on p. 138, and is, in fact, well known. What novelty there is lies in the manner in which the principle is applied.

Currents measured by an ammeter A are passed through the electrolyte by means of main electrodes p and q. The resulting P.D.s between the ends of a tube containing the electrolyte are measured by means of the voltmeter V and the auxiliary electrodes p' and q'. The ratio of the two readings gives the resistance.

The procedure when using direct currents requires little comment. We have merely to ensure that the voltmeter gives, to a sufficient degree of accuracy, the P.D. between the ends of the tube when a known current, measured by A, is passed through it.

For this purpose it is necessary to use a voltmeter of which the resistance is extremely great compared with that of the electrolyte and of which the capacity can be neglected in comparison with that of the auxiliary electrodes p' and q'. If an electrostatic voltmeter is used the resistance is, of course, practically infinite and the capacity, even of a very sensitive instrument, is very small compared with that of electrodes of moderate size. For example, the minimum capacity of electrodes of such metals as Hg Cu and Pt is of the order 10 mfd. per square centimetre, while that of the electrometer may be of the order 10^{-4} mfd. or less.

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* F. Krüger, "Zeits. f. Physik. Chemie," XLV., p. 1, 1903. For another aspect of polarisation capacity, *see* E. Rothé, "Ann. de Chim. et de Phys.," [8], Vol. I., p. 215, 1904. *Cf.* also Krüger, "Physik Zeitschr.," XI., p. 719, 1910.

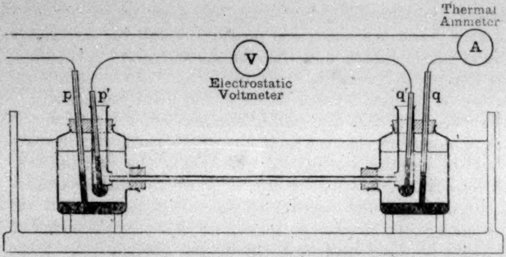
It is further necessary that the contact P.D.s between p' and q' respectively and the electrolyte should not be subject to uncertain fluctuations. For this reason it is best to use mercury electrodes whenever possible. The two contact P.D.s will then usually be not only constant but also of practically identical magnitude. Any appreciable difference between them would, of course, affect the measurements. It can be allowed for, if present, by taking the mean of the voltages obtained when the current is passed through the electrolyte first in one direction and then in the opposite. In making this latter test it may be necessary to allow for certain peculiarities of the electrometer which will be described later.

Suppose now that alternating currents are being used. It is again necessary that the capacities of the auxiliary electrodes should be very large compared with that of the electrometer; but, in addition, the quantity $2\pi^2n^2C^2R^2$, where n is thefrequency of alternation, C the capacity of the electrometer and R the resistance of the electrolyte, must now be small in comparison with unity. This expression shows that the capacity of the electrometer may require to be taken into account if the frequency and the resistance are high. For instance, if the frequency is of the order 2,500 ∾ per second and the resistance of the order 10,000 ohms, the capacity of the electrometer must be allowed for if it exceeds 2×10^{-4} mfd. when an accuracy of the order 0·05 per cent. in the measurement of R is desired.

In our case it was advisable to work with fairly high resistances because the ammeters which we had gave a much wider range of sensitivity than the voltmeter. Other things permitting, it is an advantage to use high resistances. End corrections may then be made inappreciable and, working with small currents, it is possible to reduce the disturbance due to the heating effects of the latter to a very small quantity. Further, it is relatively easy when working with continuous currents to keep these steady for a considerable time, when they are small, without using main electrodes of large area.

The current-measuring instruments at our disposal were a Duddell thermo-ammeter suitable for currents of the order 10 milliamperes and a Duddell thermo-galvanometer of which the sensitiveness could be varied within wide limits. The available voltmeter was an Ayrton-Mather electrometer (constructed by Paul) of which the deflection could not conveniently be read, with an accuracy of 0·1 per cent., unless an E.M.F. of the order 10 volts was applied between its terminals. Its capacity was, however, sufficiently small to be negligible when using the instrument in parallel with a resistance of 10,000 ohms at the highest frequency of alternation available.

We had three convenient means of obtaining alternating currents, namely, a rotating commutator with which any frequency between 0 and 100 ~ per second could be obtained, a Crypto alternate current generator giving conveniently 100 ~ per second, and a Siemens high-frequency alternator giving frequencies of 500 to 2,300 ~ per second. The latter were measured by means of a Frahm vibrating-reed frequency indicator.



In the experiments to be described the electrolyte was contained in a tube, of about 1 mm. internal diameter and about 30 cm. long, connecting two bottles in which were the main and auxiliary electrodes as shown in the figure above.

The area of each main electrode was about 30 sq. cm. Each auxiliary electrode had an area of about 2·5 sq. cm. and was contained in a test tube perforated at the side, supported as shown in the figure. This apparatus was immersed in an oil bath by which sudden changes of temperature were prevented.

The circuit was made as simple as possible, to avoid unnecessary self-induction and capacity, and contained only the electrolytic resistance, the measuring instruments and the source of E.M.F. Any desired change in the intensity of the alternating current was made by altering the magnitude of the exciting current in the generator.

The apparatus was arranged so that observations with continuous currents and with alternating currents of different frequencies could be taken one after the other in rapid succession. Readings were taken alternately with continuous and alternating currents for a period of half an hour so that a comparison between the two should not be affected by small changes of the galvanometer zero (which could not be entirely eliminated) or by a change in temperature. The continuous current was reversed between successive determinations. The current was adjusted approximately to the same value for each determination of the same series and several series with different currents were obtained. One series of readings is shown in detail in Table I. The temperature of the oil bath remained at 18·1°C. throughout the experiment. The electrolyte used was a normal solution of potassium chloride.

TABLE I.

Time.	Frequency, ∾ per sec.	P.D., volts.	Current. amperes.	Resistance, ohms.		
				0	100 ∾	500 ∾
1.50½ p.m.	500	10·235	0·001127	...	...	9,081
1.51 ,,	0+	10·115	0·001116	9,064		
1.51½ ,,	100	10·07	0·0011085	...	9,084	
1.52½ ,,	0−	10·11	0·001115	9,067		
1.53½ ,,	500	9·805	0·001079	...	...	9,087
1.55 ,,	0+	9·95	0·001097	9,070		
1.56 ,,	100	10·015	0·001102	...	9,088	
1.56½ ,,	0−	10·12	0·0011155	9,072		
1.57½ ,,	500	9·955	0·0010955	...	...	9,087
2.0 ,,	0+	9·87	0·001088	9,072		
2.1 ,,	100	10·015	0·001102	...	9·088	
2.1½ ,,	0−	10·125	0·0011165	9,069		
2.3 ,,	500	9·865	0·001079	...	...	9,087
2.4 ,,	0+	10·02	0·001105	9,068		
2.4½ ,,	100	10·07	0·001109	...	9,080	
2.5 ,,	0−	10·105	0·001115	9,063		
			Means ...	9,068	9,085	9,085

It will be seen that the greatest difference between individual readings at each frequency does not exceed 0·1 per cent. and that the mean readings for each frequency are identical within

0·05 per cent. at least. The conditions of working were such that we could expect to be able to detect differences of this order. We aimed at getting identical readings of the ammeter in successive experiments at different frequencies and comparing the corresponding readings of the voltmeter. Since we could compare these with certainty within 1 part in 1,000 and since the deflection varies approximately as the square of the voltage we could expect to detect any variation of the voltage exceeding 1 part in 2,000. Thus we could usually compare the apparent resistances to within 0·05 per cent. without difficulty, and still more nearly under the most favourable conditions (absence of zero variations).

From the numbers in Table I. there can be no doubt that the resistances as measured by this method are identical at frequencies of 100 ~ and 500 ~ per second respectively.

We obtained a precisely similar result on comparing the resistances at frequencies of 500 ~ and 2,300 ~ per second. This is shown in Table II., in which two sets of observations similar to those of Table I. are summarised, the comparative resistances being given to the nearest 5 in 10,000 in each case.

TABLE II.

Approx. P.D. volts.	Approx. current, amperes.	Resistance.		
		0	500 ~	2,300 ~
10	0·001	9,425	9,440	9,440
13	0·0014	9,395	9,410	9,410

There was, therefore, no ambiguity about the results of the experiments with alternating currents. But when these are compared with the results for steady currents it will be seen that the latter are uniformly about 1 part in 600 lower. So that even if we assume that the experimental error may amount to 0·1 per cent. there is still a small difference of at least 0·05 per cent. which remains unaccounted for.

The easiest way of testing whether this difference arose out of the method of calibrating the instruments was to replace the electrolyte by a metallic resistance and to observe whether the same discrepancy recurred. For this purpose a resistance of about 700 ohms was constructed by depositing a thin film of gold upon a strip of glass.

It was now necessary to use currents of the order 0·014 ampere to get P.D.s of the order 10 volts. The results of two series of observations are shown below :—

TABLE III.

——	Frequency.			
	0	100 ~	500 ~	2,300 ~
Resistance I. ...	734·9	...	736·5	736·4
Resistance II....	733·4	734·9	735·0	...

Similar results were obtained with various other resistances. Differences of exactly the same order as those obtained with the electrolyte were always observed. (We found, using different steady voltages, that the resistance of the gold strip was always the same at the same temperature within the limits of experimental error.)

From these results it is obvious that the small differences exhibited in Tables I. and II. do not represent any peculiarity of the electrolyte and that the resistance of the latter was the same within 1 part in 2,000 whether it was carrying steady currents or alternating currents of any frequency up to 2,300 ~ per second.

III. *Supplementary*.

It is perhaps desirable, in view of the behaviour of the instruments, that the method of calibration should be indicated.

The ammeter gave the same deflection when a known current was passed through it in either direction and the readings showed that the couple producing deflection of the suspended system was very nearly proportional to the square of the current. We therefore assumed that when an alternating current i, of period τ, produced the same steady deflection as a continuous current i_0, the relation given by the equation

$i_0^2\tau = \int_0^\tau i^2 dt$ was true.

The voltmeter, however, gave appreciably different readings when the same E.M.F. was applied to it in different directions. In our calibration, upon which the results given in the Tables depend, we assumed that this was due to a contact P.D. within the instrument which remained constant during the measurements. Calling this P.D. e we applied a known steady voltage v_0, first in one direction and then in the other, and assumed that the two deflections θ_1 and θ_2 could be written

$$\theta_1 = k_{\theta_1}(v_0 + e)^2 \text{ and } \theta_2 = k_{\theta_2}(v_0 - e)^2$$

respectively.

Since θ_1 and θ_2 were nearly equal and since we had reason to believe that $dk_\theta/d\theta$ was small, we took $k_{\theta_1}=k_{\theta_2}=k$. Whence we got $\frac{1}{2}(\theta_1+\theta_2)=k(v_0^2+e^2)$. We then assumed that if we applied an alternating voltage v, of period τ, satisfying the condition $\int_0^\tau v^2dt=v_0^2\tau$, we should get a steady deflection $\theta=\frac{1}{2}(\theta_1+\theta_2)$, since the mean couple tending to produce deflection in this case would be proportional to

$$\frac{1}{\tau}\int_0^\tau (v+e)^2dt, \text{ i.e., to } v_0^2+e^2.$$

We have found, however, that a small error, which would produce differences of the same sign as those shown in Tables I. and II., can be introduced in this way. Because, even if it be correct to explain the difference between θ_1 and θ_2 by means of a contact E.M.F., it is possible to show that it is incorrect to assume that this E.M.F. is constant. In the instrument we used it decreased perceptibly when v was increased. For example, applying first a low steady voltage v_0, we inserted between this and the electrometer a small constant E.M.F. of such sign and magnitude that the voltmeter deflection remained unchanged when v_0 was reversed. The apparent contact E.M.F. was then balanced. Increasing the applied voltage until the deflection was approximately doubled it was easy to see that the contact E.M.F. was now over-compensated by the inserted voltage. An approximate estimate showed that, in one case, the apparent value of e decreased from about 0·07 volt when the applied voltage was 7 to about 0·06 when the applied voltage was 10.

Hence e is a function of the applied voltage. Its rate of variation may depend upon the sign of the latter; but unfortunately its relative magnitude, compared with the voltages required to produce accurately measurable deflections, was so small that it was impossible with our instrument to decide this question. For obvious reasons we are unable to say whether these peculiarities are shared by other instruments.

ABSTRACT.

Some experiments upon this question were exhibited before the Society by Dr. Haworth in 1911. In these a modification of Wien's method was used—the optical telephone being replaced by a vibration galvanometer—and the conclusion was drawn from them that the resistance of an electrolyte varies to an easily perceptible degree with the frequency of the alternating currents to which it is subjected.

The authors point out that the terminal difference of potential and the current in a branch of a network containing capacity and self-induction may be in the same phase although their ratio does not give the resistance of that branch. This will happen (whether the branch contains an electrolyte or not) if there is leakage through the condensers, causing the P.D.s between their plates to be out of quadrature with the current. The apparent resistance of the branch will then be a function of the frequency of the alternating currents circulating in the bridge, and Wien's method (as Wien himself points out) will give this apparent resistance only.

It is, therefore, unsound to use the method to test whether the resistivity of an electrolyte depends upon the frequency of the currents to which it is subjected, unless it is shown that the effects of leakage through the electrolytic condensers can be neglected or allowed for.

A particular case was indicated by the authors, in which the leakage could be made large or small at will. The results for this case had been interpreted by Krüger, without assuming any variation of resistance with frequency, in a manner which seemed satisfactory to them.

In order, however, to remove or justify any doubt upon the question they have performed test experiments by a simple and direct method which was described. It depends upon simultaneous measurement of the voltage between the ends of a tube containing the electrolyte and of the current passing through it. The former was measured by means of an Ayrton-Mather electrostatic voltmeter connected to auxiliary electrodes and the latter by means of a Duddell thermo-galvanometer.

In the cases examined it was found that the resistivity of the electrolyte was constant within 0·05 per cent., whether steady currents or currents of any frequency up to 2,300 alternations per second were used.

Until the instruments were calibrated by means of a metallic resistance there appeared to be a small difference of about 1 part in 600 between the resistance as measured by continuous currents and the values obtained with alternating currents.

Some supplementary experiments were made with the object of elucidating the peculiar behaviour of the instruments which this calibration disclosed. On account of the smallness of the effect its cause could not be completely ascertained; but the fact that the apparent contact P.D. within the voltmeter was a function of the applied voltage, decreasing as the latter was raised, would cause an effect of the same sign as that observed. Unallowed-for leakage, greater with steady than with alternating currents, might also provide a partial explanation of the results.

DISCUSSION.

Mr. A. **Campbell** expressed himself interested, and thought that the very small discrepancy described might be due to the method employed, which involved taking direct readings of both volts and amperes. He had not noticed the effect himself. It might be that it only occurred in particular instruments. He thought the effect was more likely to lie in the thermo-ammeter than in the electrostatic voltmeter.

Mr. F. E. Smith asked how long the current was left on before readings were taken, and was it possible that with direct currents the effect might be due to differences of concentration round the electrodes. He agreed, however, that the fact that gold showed the same phenomena seemed to negative this possibility.

Mr. R. S. Whipple suggested a radio-active leak in the voltmeter.

Prof. G. W. O. Howe thought the discrepancy was too regular to be due to any radio-active leaks.

Dr. H. F. Haworth remarked that Dr. Smith in his equations had assumed R to be independent of the frequency. He suggested that the resistance decreased as the frequency was raised because the complex molecules in the solution were shaken asunder, and also suggested that using platinised electrodes instead of ordinary platinum might have the effect of disintegrating these complex molecules in the solution, thereby giving a constant value of the resistance. The mercury electrodes employed by Dr. Smith might also have just the same effect. Heating the solution also decreases the resistance by breaking up the complex molecules.

Mr. G. D. West asked if the authors had used lower frequencies than 100.

Mr. G. L. Addenbrooke remarked that it had been known to him for a long time that contact P.D. was by no means constant. It could be altered by fingering the apparatus. He thought the deflections with alternating currents might be influenced by vibration of the electrometer needle.

Mr. F. P. Worley remarked that physical chemists were entirely dependent on physicists for the method they employed in measuring electrolytic resistances, and any suggestions for improving methods followed by chemists would be very acceptable to them.

Prof. C. H. Lees remarked that Dr. Smith had given a very simple explanation of the facts, and it was open to Dr. Haworth to explain them in any other way if he preferred.

The Authors replied briefly to the various questions asked and criticisms offered as follows: (Mr. Campbell) The observations show that the discrepancy, although very small, lay outside the limits of experimental error. It would be interesting to examine the behaviour of other instruments of the same and of different types. (Mr. F. E. Smith) [Readings were taken continuously for half an hour or more, direct-current readings in opposite directions being made between those with alternating currents. The direct-current readings showed no systematic differences such as would arise from concentration changes. The latter would be produced at the main electrodes only, and could not affect the readings except indirectly by diffusion, of which the effects would be negligible in the apparatus used. (Mr. Whipple and Prof. Howe) The variation of the apparent contact P.D. with the voltage might have an electronic origin. (Mr. West) There was the same difference between the continuous and alternating-current effects even when the frequency was the lowest (about 10 ~ per second) that would give steady readings. (Mr. Addenbrooke) The apparent contact P.D. for a given voltage, in the electrometer used, changed very little from day to day. The variation with the voltage, to which they referred, was a different phenomenon. If the vibration of the needle produced any effect they would expect that effect to be different at different frequencies, but no evidence of this was found. (Mr. Worley) Kohlrausch and Holborn had given information sufficient to satisfy all ordinary needs in their "Leitvermögen der Elektrolyte."

Dr. Haworth was, of course, entitled to seek fresh explanations if those of other workers seem to him insufficient. In order to find whether the resistance of a conductor, metallic or otherwise, varies with frequency, it is necessary to avoid possible sources of error in the method of measurement. With molecules in solution possessing the properties he suggests it would be remarkable if Ohm's law were obeyed by electrolytes under any circumstances. His next remark suggests that he has found since his first experiments with platinum that the apparent variation with frequency can be reduced by platinisation. To the authors this result presents no difficulty. If, however, Dr. Haworth's explanation is correct, new results should appear. The apparent resistance of an electrolyte should be alterable by merely placing pieces of platinised platinum in the containing vessel. Similarly, the freezing point or the boiling point of the electrolyte should be altered by the addition of this material. So also with mercury. With regard to the effect of heat, the opinion is very generally held that most of the results of other observers are best explained by making the opposite assumption to that which Dr. Haworth favours.

XIV. *The Dynamics of Pianoforte "Touch."* *By Prof.* G. H. BRYAN, *Sc.D., F.R.S.*

RECEIVED FEBRUARY 6, 1912.

THE problem of obtaining formulæ for the vibrations of a stretched string when struck by a pianoforte hammer is approached from two very different points of view by Helmholtz and Kaufmann, and there is a correspondingly wide difference in the conclusions to which the investigations lead.

Helmholtz starts with the assumption that the force exerted by the hammer on the string is given by an expression of the form $F \sin pt$, lasting from $pt=0$ to $pt=\pi$, and he applies Fourier's method to determine the amplitudes of the various vibrations set up. While with p constant these amplitudes are all proportional to F, the maximum value of the exciting force, a variation in the value of p will manifest itself in a change in the *quality of tone* as dependent on the relative intensities of the fundamental tone and its harmonics.

Kaufmann, on the other hand, bases his investigation mainly on the *functional* solutions of the differential equation of wave propagation along a stretched wire. He thus obtains a differential equation for the motion of the hammer. In his analysis he neglects all such forces as those due to the weight of the hammer, and thus arrives at the conclusion that if the string is indefinitely extended in both directions the hammer will never leave it—a conclusion which ought to suggest the obvious inference that it is not justifiable to leave out the weight of the hammer and similar forces without further investigation. Kaufmann's assumptions necessarily lead to the conclusion that so long as the same string is struck by the same hammer at the same point the *quality of the sound* emitted will always be the same. A general discussion is, however, given as to the influence of elasticity of the hammer on the resulting quality of tone as determined by the relative intensities of the harmonics.

Now there are good reasons for believing that the *quality* of the tones produced when playing the piano is capable of being varied not only, sometimes, by playing louder or softer but also by other differences in playing commonly associated with the somewhat ill-understood term "touch." The difference between the delicate or brilliant touch of the professional pianist and the dull heavy thumping of the schoolgirl may doubtless be accounted for to a large measure by judgment or want of judgment in the loudness with which the notes are struck, proper or improper use of the loud pedal, small deviations from uniformity in the lengths of different notes or the presence or absence of ornaments on the top of the piano. But, in addition to all these differences, it will probably be generally admitted that there are other differences which seem to be due to the varying *elasticity* of the performer's fingers, and it is certain that many people training as pianists attend to the physical development of the muscles of their hands to an extent that would hardly be necessary if it were merely a question of regulating the total strength of the blow delivered to any key of the pianoforte.

The growing popularity of pneumatic piano-players used in conjunction with perforated rolls renders it desirable that the dynamics of pianoforte touch should be examined in greater detail than has been done either by Helmholtz or by Kaufmann. It is very generally admitted or assumed that there is a certain element missing in the performance of these machines, namely, the touch of the human fingers; and certainly the average piano-player of commerce as usually played has very little resemblance whatever to the interpretation of the professional pianist. Indeed, as applied to piano-players in general, the unfavourable opinion expressed in the "Encyclopædia Britannica" is quite a fair criticism.

A good piano-player in the hands of an experienced performer certainly cannot be described as a *mechanical musical instrument*. But it will often be found that if the keys of the pianoforte are struck with the fingers, tones are emitted which cannot be exactly reproduced by any manipulation of the piano-player, however skilful. The difference seems to be in the *quality* of the sound rather than in its loudness or softness, and if this view is correct, Kaufmann's theory is not sufficient to account for the difference.

Now there are many people who for their enjoyment of good music have to depend largely on getting the greatest possible range of effects out of a first-class piano-player, and it thus becomes interesting to examine more closely the extent to which *differences of touch can be made to cause differences in the quality of individual notes.*

The existence of any such effect obviously suggests the assumption that the impressed forces acting on the hammer so far from being negligible, as Kaufmann assumes, are capable of considerably modifying the action of the hammer during the short interval of time that it is in contact with the string, and one such modification would be a lengthening or shortening of the duration of contact. If the differences are really independent of the loudness of the note, they can only be controlled by the pianist if he is able to produce time-variations of the pressure of his finger on the keys of the piano while they are being depressed, and it is certain that such time-variations would naturally result from variations in the elasticity of the performer's fingers, such as could be caused by a tightening or slackening of the muscles. At the same time it is very difficult to find a satisfactory explanation on dynamical principles of the relation between cause and effect, and in this connection the dynamics of pianoforte touch presents greater difficulties than the corresponding investigation for other musical instruments, the reason being the shortness of the time-interval during which the notes are excited. It is easy enough to see how the character of the tones of a violin might be altered by varying the pressure or velocity of the bow, or the elasticity of the hand drawing it, and it would not be hard to reproduce these variations in a purely mechanical experiment.

In the case of a well-adjusted horizontal grand piano, if a key is slowly depressed the hammers can be made to nearly touch the wires. If the hammer actually press against the wires the note is said to be "blocked," and fails to sound properly. It does not follow that while the hammer is touching the string no forces are impressed on it through the check action, but it does follow that these forces are not sufficiently great to support the weight of the hammer since its position of equilibrium is below the string, and the view that their effect is appreciable is directly contrary to Kaufmann's hypothesis.

In a subsequent communication I propose to investigate the differential equations of motion of the hammer in the more general case assuming various different expressions for the impressed forces acting on it. Before doing so I propose to describe a simple apparatus by means of which differences of pianoforte touch can be readily reproduced and the dynamical actions associated with them made the subject of qualitative observation.

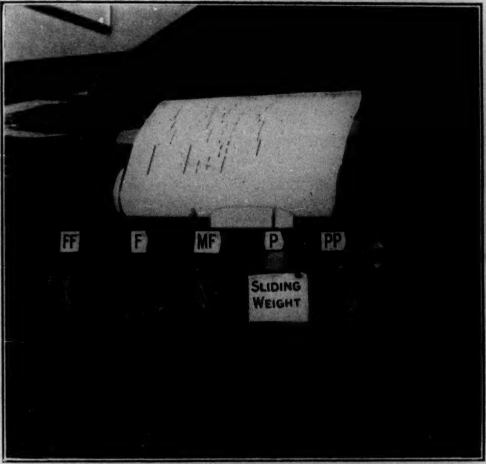
This apparatus (Fig. 1) consists of a long horizontal lever fixed in front of the usual levers of a pneumatic player of the ordinary "Standard" type, and operating directly on the auxiliary regulating bellows or wind chamber. This lever carries a sliding weight (about 12½ oz.) and supersedes the ordinary spring. While in the commercial player, the collapse of the auxiliary bellows is regulated only by the elasticity of the spring, which remains constant, the collapsing tension can now be varied in several ways. It is reduced by shifting the sliding weight to the right, increased by shifting it to the left, and thus it is possible to produce very rapid time-variations of the pneumatic tension reproducing the conditions that prevail when a pianist strikes the keys in different ways apart from the mere question of loudness or softness in playing. These variations are further accentuated by applying hand-pressure directly to the lever, thus causing the actual touch of the human hands to govern the variations of pneumatic tension and so to be transmitted directly to the keys of the piano.

It is well worth while to take a good deal of trouble in trying different ways of manipulating such a lever and comparing the results with the effects produced by a good pianist, as in this way a great deal can be learnt about the dynamical actions employed in producing these effects. After a little such practice it is interesting to disconnect the lever and replace the usual spring.

If my experience is any guide, the result will be disappointment, and the effects previously regarded as satisfactory will be found to be lacking in many respects. Indeed, after a very rough-and-ready preliminary experiment with sticks and strings and a kitchen weight, I found it impossible to obtain the same satisfaction with the old arrangement that I had experienced for 10 years previously.

The advantage of this arrangement is that it enables the dynamical actions to be studied by the same person who ordinarily uses the piano-player and it thus eliminates differences due to individuality which render difficult a comparison of the results produced by *different* performers, one operating with fingers and the other employing the modern pneumatic mechanism.

By merely setting the sliding weight against the letters corresponding to the expression marks in the music, the result is increased breadth of contrast, the soft passages being less dull and the loud ones less thin. By moving the sliding weight independently of the pedalling, variations of *touch* are at once obtained bearing a close resemblance to those of the professional pianist. With the weight at *pp* and strong pedalling,


FF
F
MF
P
PP
SLIDING WEIGHT

[Patent applied for.]

APPARATUS FOR CONTROLLING TOUCH IN A PNEUMATIC PIANO PLAYER.

To face page 150.]

a very "metallic" effect is produced and the treble parts predominate over the bass. With the weight at *ff* and light pedalling, softer and richer effects are produced, the bass parts now coming into prominence. In this way it is interesting to attempt a pianoforte rendering of organ music and to produce contrasts suggestive of a two-manual organ.

With regard to what can be done by hand control the "metallic" tone is greatly increased by depressing the lever and allowing it to sharply fly up again, whereas the vigorous bass effects are improved by heavily depressing the lever. A sudden jerk up from below in a loud passage often saves a treble note from being drowned by the bass.

The most noticeable feature of these experiments is the very marked differentiation that it is possible to produce between notes in different parts of the keyboard. This differentiation is particularly noticeable in the case of long chords extending over a considerable length of the keyboard; in such cases the intensities of the treble and bass parts can be varied quite independently of each other. But even in comparatively short chords it is often possible to appreciably accentuate certain notes of the chord and to vary the effect at will.

These results are due to the fact that different notes of the keyboard require different kinds of touch to bring out their maximum effect, these differences depending on time variations of the pressure applied to the keys of the piano. A short, sharp pressure produces its greatest effect at the treble end leaving the bass notes unaffected; a long, heavy pressure brings out the bass notes without affecting the treble; indeed, if we use the term "exposure" to denote the time interval during which the pressure is effective in exciting any particular note, it will be found with a little experience that the correct exposure varies continuously from one end of the keyboard to the other. It is possible with practice to judge exactly the exposure necessary to give prominence to notes in any desired part of the scale, and thus to reproduce the great majority of effects which a good professional pianist produces in accentuating particular notes of chords. To obtain the maximum enjoyment out of a piano-player it is necessary to know the feel of the touch of every note on the keyboard and this is by no means so difficult as it appears.

The actual "exposures" in the case of a piano-player are not so short as might be supposed. I have frequently failed to bring out notes in the lower part of the treble clef through under exposure. In the case where the touch is controlled by the sliding weight it often suffices to place this opposite the part of the music roll on which are the notes which have to be accentuated.

An explanation of these effects is doubtless largely to be sought in the varying masses of the pianoforte hammers, which cause them to undergo different accelerations when the same force is applied to them by a common pneumatic pressure (or rather tension). The lighter hammers rise more quickly, and when they are released it will make all the difference whether the heavier ones are firmly pressed up or the pressure on them is suddenly reduced.

It still remains, however, problematical whether the effects are entirely due to this cause or whether the length of time that the hammers are in contact with the strings does not undergo a corresponding variation, in which case there will be a certain duration of contact that will give the maximum effect to a particular note as is provided for on Helmholtz's but not on Kaufmann's theory. Further, it is hardly possible by playing *chords* to judge of the existence or non-existence of differences in the quality of individual notes. For this purpose it is necessary to select a passage containing single notes only. The test is difficult to perform even with the present apparatus, for several reasons. When notes are struck loud the *impression* is certainly produced that the harmonics are more brilliant as a general rule. But this effect might be due to peculiarities in the sense of hearing. On the other hand, it is not easy to vary the action without altering the loudness, and it is very easy to fail. To some people a note struck on the piano sounds louder or softer and nothing else. On the other hand I have found that most people notice marked differences of quality in the notes of a solo passage produced by shifting the sliding weight on the lever although some of them state they have no ear for music.

Personally I should have not the slightest doubt on the matter so far as it is capable of being decided by observation. Indeed, I would almost be prepared to go so far as to say that piano playing, whether by hand or pneumatic action, must necessarily sound rather dull and mechanical unless individual control is capable of influencing the quality as well as the intensity of particular notes. But when we attempt to give a theoretical explanation, to apply the principles of dynamics and to write down equations of motion, difficulties occur.

Taking the case of the horizontal grand piano, we have seen that if impressed forces act on the hammer during its contact with the wire, these forces must be less than the weight of the hammer. In order to form an estimate of the effect of such forces, I have modified Kaufmann's equations by supposing the hammer to be subject to the downward acceleration of gravity, taking $g=981$. The results at first appear very promising, and they indicate that, in the case of a string struck near the middle, gravity may have an important effect on the duration of contact. In the case of a string struck near one end, the effect of gravity is much more limited. After a wave has had time to travel from the hammer to the near end and back again, the terms depending on gravity disappear from the equations of motion, and the initial effects appear, so far as the calculations go at present, to be far too small to produce an appreciable result. It is true that the striking velocities assumed by Kaufmann are largely in excess of those required for artistic piano playing. I find that for loud passages a striking velocity of 150 cm./sec. is quite sufficient, for soft passages 30 cm./sec. is more than enough ; such a velocity would be just sufficient to carry the hammer 5 mm. above the level of the wire under gravity. For really delicate playing an even smaller velocity is needed. Nevertheless, the equations do not at present seem likely to afford the required explanations. In the case of the vertical piano, the position of equilibrium of the hammers is much further from the strings and the effect of gravity much smaller. It is clear that we must either abandon our belief in the effects of " touch " on " quality of tone " or seek an explanation of a more general character.

The case is somewhat different when the elasticity of the hammer, where it touches the wire, is taken into account. From Kaufmann's extreme case of an inelastic hammer, we may pass to the opposite extreme of an elastic hammer striking a fixed rigid wire. In this case the expression for the force will assume Helmholtz's form. If, however, the hammer be also acted on by a downward acceleration due to gravity the expression for the exciting force will assume the form $F \sin pt -$ constant, so that the duration of contact will be reduced and will become shorter as the striking velocity is decreased.

It should be mentioned that some of the inferior piano-players and music rolls are not well suited for showing differences of pianoforte touch. Many such instruments have separate treble and bass buttons, so that it is difficult to get a continuous differentiation between various parts of the scale. Others have large springy reservoir bellows, which collapse when a loud effect is required, and come smashing out on the soft passages which follow. In many music rolls, too, the chords are badly ranged. If the treble notes are even $\frac{1}{50}$ in. before or behind the bass ones, as is often the case, any attempt at differentiation is almost certain to fail and the usual dull mechanical playing is the result. Apart from these difficulties, there is an essential difference between sheet music and music rolls, the former being divided into bars, with the number of beats in each bar indicated at the beginning, while the latter possess no such indications. This feature alone renders it impossible for a person without a thorough knowledge of the music to rival the pianist of even limited experience.

Most modern inventions, such as the aeroplane, motor car, kinematograph and gramophone have been the subject of discussion in scientific and technical journals, but hitherto the pneumatically played piano has been conspicuous by its absence, So far as my experience goes there are, however, a good many people who are quite willing to take a little trouble in extending their power of control over these instruments, and under these circumstances a discussion on the dynamics of pianoforte touch may not be altogether out of place in a scientific body like the Physical Society.

ABSTRACT.

The author discussed Helmholtz's and Kaufmann's theories of the vibrations of a pianoforte wire excited by impact, with special reference to the effects obtainable with the modern pneumatical piano-players and player-pianos, and the common widespread belief that these can never reproduce the touch of the human fingers. While the rendering of many commercial piano-players in the hands of an average performer bears little resemblance to the performance of a professional pianist, the author finds that there is generally believed to be a certain element missing even in music played by a skilled performer on a first-class modern piano-player, this missing element being commonly associated with what is described as "touch." In view of the great value of piano-players to lovers of music, it thus becomes interesting to examine more closely what is meant by "touch," and whether it is capable of being reproduced to a greater extent than hitherto upon pneumatically controlled pianos.

The question turns very largely on the extent, if any, to which the *quality* of individual notes can be varied by striking the notes in different ways. Such a possibility involves the inferences that (*a*) the intensities of the fundamental tone and its several harmonics are capable of independent variation ; (*b*) these variations can only be produced by varying the behaviour of the pianoforte hammer while

it is in contact with the string, for example, by lengthening or shortening the duration of contact; (*c*) such an effect can only be produced by rapid time variations of the pressure applied to the keys while they are being depressed—*e.g.*, by a fairly rapid decrease or increase of pressure produced by smartly striking or heavily pressing on the key.

The author finds that Kaufmann's investigation fails to account for any such effects, and that difficulties arise even when the equations are modified so as to take account of impressed forces on the hammer comparable with those due to gravity. On the other hand, he describes experiments which appear to indicate beyond all reasonable doubt the existence of such effects of "touch," and which certainly demonstrate the possibility of reproducing them by means of the modern "pneumatic" instrument. For this purpose the author's piano-player, which is a first-class instrument of the "Standard" type, but *with the whole keyboard under one common control*, was fitted with an "auxiliary lever" for which a patent application has been filed. This lever operates directly on the face of the auxiliary regulating bellows, and the air-tension in the bellows can be regulated by means of a sliding weight placed on the lever, or by applying hand pressure to the lever itself. In this way *the touch of the human hand can be transmitted directly to the keys of the piano*. So far as the experiments go, they indicate that even if the lever is worked in conjunction with suitable expression marks, as could be done by a person of moderate experience, increased breadth of contrast is obtained. While by varying the position of the load independently of the pedalling a variety of dynamical effects can be produced, which can further be increased by hand control.

A short, sharp pressure produces a bright ringing treble with a light bass, a sustained pressure produces a rich bass with a soft treble; the general character of the tone being suitably described as "metallic" in the first case and "woody" in the second. A very conspicuous feature of these experiments is the marked differentiation which they show between notes in different parts of the scale, especially in chords, the notes of which are accurately ranged (as is unfortunately often not the case in music rolls). The duration of the pressure required to produce the maximum effect on a particular note of the piano varies continuously from the treble to the bass end, being least in the treble and greatest in the bass, and by means of this natural or dynamical differentiation notes in a particular part of a chord at any part of the scale can be accented independently of the rest.

Whether it is possible to vary the quality of individual notes is a point that can only be tested by playing single notes as opposed to chords. The differences that can be effected can only be noticed by a trained ear; in the author's experiments it has been found that some persons notice *very marked* differences, others notice very slight differences, others no differences at all. The differences are probably as conspicuous as those between a stopped string and a harmonic on the violin. It is not always easy to produce these differences for purposes of demonstration, though it is often easier to do so in the course of playing through a suitable composition. In any case the author finds that the effects can be obtained more easily with a pneumatic player fitted with auxiliary lever than in striking the keys with fingers. When the lever is disconnected the change observed affords some indication of the origin of the popular belief in the limitations of the pneumatically played piano.

Unlike the gramophone, aeroplane, motor car and cinematograph, the modern piano-player has been conspicuous by the absence of reference to it in scientific and technical journals. The present Paper, which arises out of an attempt to obtain a closer degree of approximation to *playing the piano* with these instruments indicates that they open up some interesting problems in the study of acoustics.

DISCUSSION.

Mr. A. **Campbell**, speaking as a musician, said that Prof. Bryan's invention largely increased the capabilities of the mechanical piano-player, as it gave the person who guided the machine the power of accenting particular notes at any part of the keyboard. If the *loudness be kept constant* the differences in quality of tone that could be obtained from a piano were quite infinitesimal compared with the difference given by two pipes of different stops on an organ, and were of negligible value in musical art, although their importance was so commonly magnified by fancy and by professional humbug. Piano music had to depend for its variety on factors other than difference in tone quality. The comparative constancy in tone quality could, no doubt, be demonstrated objectively by means of phonautograph records. Referring to the general question of mechanism in relation to art, he considered that the use of mechanical players tended to reduce the number of persons who acquire a real knowledge of music by learning to read it by voice or instrument in the ordinary way, and hence such machines must inevitably prove destructive to musical art.

Dr. W. H. **Eccles** entirely disagreed with Mr. Campbell, and considered the difference was most marked, and that Prof. Bryan's lever made a great difference in the contrast obtainable; but he thought it would be an improvement to be able to make the soft passages softer.

The **President** remarked that the explanation of the difference of tone of a note according as to the manner in which it was struck—such as, for instance, between the touch of a learner and a skilled player—must interest many of even of those who were not musical.

Mr. G. H. **Berry** (communicated remarks), as a practical pianoforte maker, certainly thought Prof. Bryan's invention gave a greater control over the instrument to the gifted musician. With regard to the question of varying the time of contact of hammer with string by variations in the touch, in his opinion it was impossible to do this. When the hammer was in contact with the string the key was entirely disconnected from it, and also the *check* was merely a repetition device, and did not come into action until the hammer had left the string. The only thing that could be varied (neglecting the loud and soft pedals) was the velocity with which the hammer struck. It seemed to him that with Prof. Bryan's invention a greater control was obtained over the velocity. Kaufmann's investigation, to which Prof. Bryan referred in his Paper, neglects a point of primary importance in the pianoforte. Stokes had shown that a vibrating string, if insulated from a sound board or resonator, was inaudible; therefore the sound from a pianoforte was produced, not by the string, but by the sound board. The latter had a natural frequency of its own, and was set in vibration by the blow of the hammer transmitted through the strings. The natural frequency of the sound board was rapidly damped. (Some experiments of his own showed that it seldom gave more than about six waves, the first two or three being of larger amplitude than the forced vibrations caused by the string.) Was it not highly probable that the varying strengths of the harmonics set up by different velocities of the striking hammer were caused by the difference in the amount of energy absorbed in producing the natural frequency of the sound board?

Mr. V. **Lough** (communicated remarks) considered that as an addition to the ordinary pneumatic player Prof. Bryan's expression regulator should prove very useful. It certainly enabled a skilled operator to make a great improvement in the rendering of the music. There were, of course, other arrangements by which the pressure in these players could be varied suddenly to emphasise particular notes or chords; but Prof. Bryan was, he thought, the first to introduce the principle of variable inertia in the pressure regulator. Although the conditions of the performance were not very favourable, there was certainly a very marked difference between the "metallic" and the "richer" effects, and this he supposed meant a change of tone quality; but how much of this was due to the regulator was not clear, owing to the use of the sustaining pedal. He could, however, detect little in favour of Prof. Bryan's claim of differentiating between notes struck simultaneously, an effect which was obtainable in some players. In any case there were none of the clear singing tones and delicate touch effects which could be produced by a good pianist, but were still beyond the range of the mechanical player. As regarded the scientific aspect of the matter, the Paper did very little to justify its title. Very little information was given as to the analysis of the dynamics of the hammer stroke, and the results were only vaguely indicated. The problem of the variation of tone quality with touch was not appreciably advanced by this Paper.

Mr. T. Harding Churton (communicated remarks) stated that Prof. Bryan referred to what he described as the missing element of "touch" in the performance of the mechanical piano-player, and discussed what was meant by "touch." An examination of the mechanism of a piano made it evident that, no matter how a key was depressed, the result depended entirely upon the velocity imparted to the hammer that struck the wire or wires, and that the only further effect of holding the key down was to hold the damper off the wires, and thus allow the vibration to die down more slowly than if the damper were allowed to resume contact with the wires earlier. It was, therefore, evident that no variety of tone could be produced in a piano that depended upon the striking of the key which could not be produced by suitable mechanical means for striking the key with the required degree of force. But what was ordinarily conveyed by the term "touch" was not the tone produced, nor merely the degree of force used, but the effect produced by (1) the *relative* loudness of the notes played, whether struck simultaneously or in sequence; (2) the relative duration of the notes and intervals of rest ("staccato" or "legato"); and (3) by slight variations from the strict time of playing individual notes or group of notes. In short, the effects of "touch" depended upon the particular manner in which notes were played with respect to force and time, and the value of these were generally being continually varied in a more or less complex manner by a skilful player playing by hand. The more regular or uniform performance of the mechanical player constitutes, in fact, the characteristic difference between mechanical and hand playing. The music roll was capable of being cut so as to reproduce the playing of a pianist as regards *time*, but until the mechanical player was also provided with automatic control of the *force* with which each individual note was struck, the effect of "touch" must remain its missing element.

The Author, in reply, wrote: The fundamental question which was raised by my Paper may perhaps be best stated as follows. When the key of a pianoforte is depressed, either by the finger or by pneumatic action, is the resulting effect a function of one variable only (namely, the striking velocity), or is it a function of other variables as well? The assumption is, of course, made that certain other conditions remain the same in all cases. For example, the sustaining pedal must either be held down in every case, or, if this is not done, the note must always be held down for the same length of time after it has been struck. In adopting the latter method the experiments can be more easily made with the pneumatic player, since it is not easy to strike a key sharply with a finger and at the same time hold it down sufficiently to keep the damper from descending. Now I entirely agree with Mr. G. H. Berry that the construction of a pianoforte renders it very difficult to accept anything but the one-variable theory; at the same time I am bound to state that I find it impossible to disbelieve in the two (or more) variable hypothesis, in view of results which I have been able to obtain on a piano, with a piano-player in a room, with all of which I have become familiar after 10 years' experience. The discussion at the Physical Society afforded some opportunity for an inquiry into this paradoxical question; at the same time I have made numerous inquiries in other directions, and I find that there is a widespread belief in what I have called the two-variable theory. Further, I am told that in Germany, in teaching the pianoforte, this theory is tacitly assumed, two different kinds of touch being distinguished as apart from the mere differences of loudness or softness. On the other hand, it is a natural corollary of the "one-variable" theory that when the sustaining pedal is held down it makes no difference whether notes are played "staccato" or "legato," yet a great many pianists are very careful to maintain this (under such circumstances unnecessary) distinction. Some important light has been thrown on this matter by Mr. W. H. Gray, B.Sc., of Bangor, a research student in chemistry who is also a first-class pianist, and with whom I recently had a conversation on the subject. Unfortunately, he was unable to give me any references to the published Papers on the subject which he had read. His conclusions were, however, somewhat as follows: (1) It is possible to vary the quality of a pianoforte note, independent of its loudness or softness, to an extent which has a very appreciable influence on the musical effect. (2) A pressure strong at first and rapidly decreasing as the key descends produces a brilliant tone rich in harmonics. (3) A pressure gradually increasing as the key descends produces a softer tone in which the harmonics are subdued. (4) *It has been proved experimentally* that these differences are associated with differences in the length of time that the hammer remains in contact with the string. As in Kaufmann's Paper, the striking part of the hammer was covered with metallic foil, and the time of contact determined by carrying an electric current through the wire and hammer. (5) *These differences are due to vibrations set up in the rod of the pianoforte hammer.* Conclusions 1, 2, 3, are identical with the results which I had obtained independently by means of the piano-player, and which were unknown to Mr. Gray at the time he stated them. With regard to (4), it is desirable to obtain further information about these experiments. The explanation (5) seems rather improbable at first sight, but there are several points in its favour. The force applied to raise the hammer is, owing to the short leverage, large compared with its weight, and the bending moment set up near the end is considerable. The vibrations, doubtless, die out rapidly, but the interval between release and striking is very small indeed. Similar vibrations can be easily visualised on a large scale in a billiard cue. Mr. Berry refers to one vibrating system that has not been taken into account—namely, the sounding board. I think under the circumstances that the vibrations set up in the hammer may be equally important. An object lesson can be obtained by experimenting with a wine glass, in which the first harmonic is usually stronger than the fundamental, but by suitably striking the glass either may be made to predominate at will. Mr. Lough seems to have overlooked the fact that the Paper was only the first part of a more general investigation of the dynamics of the pianoforte hammer, and, in view of the evidence now elicited it is most fortunate that the second part, to include the actual equations of motion, was not published, as it would certainly not have taken account of some of the conditions which are now shown to be necessary. I now propose to defer the matter still further, pending inquiries as to the experiments referred to by Mr. Gray. With regard to the question of obtaining delicate *pianissimos* with a piano-player, as referred to by Dr. Eccles, the illustrations were evidently very unsatisfactory in this respect, owing to a mistake in estimating the acoustic properties of the room. In order to obtain the highest possible effects it is of course necessary (1) that the piano and player should be in perfect adjustment; (2) that the springs of the bellows should be of the right strength, which is usually far from being the case, and here the auxiliary lever comes in useful; (3) that the performer should be thoroughly accustomed to his instruments; (4) that the piano should be kept open, otherwise the higher harmonics are absorbed; (5) that the performer should concentrate his thoughts on the music, performing the necessary manipulations instinctively and not consciously; (6) that he should be in good form at the time, this being perhaps the most important item of all. In comparing pneumatic with finger playing in this matter I consider that the possibilities are certainly not unfavourable to the pneumatic player. It appears by no means easy for an ordinary pianist to obtain the most delicate effects, and very few of the professional and other performers at the concerts of our local musical club obtain such good results as are certainly *obtainable* with a piano-player. As regards the few exceptional pianists who obtain such marvellous *pianissimos*, it would be necessary, in order to institute a satisfactory comparison with their results, that a performer should have given as much time and attention to practising the piano-player that they have given to practising the piano. An amateur using his piano-player only in leisure half hours cannot expect to get anywhere near Paderewski. It would be undesirable, however, to digress much further on the subject of piano-players apart from the dynamical and acoustical properties suggested by them. But, judging from Mr. Campbell's remarks, I think it is rather a pity that he does not use a piano-player, as he would find that great charm and interest is afforded by these attempts to analyse pianoforte touch, and to utilise the results in the interpretation of classical music.

[It has since been proved that the absence of delicate effects in the illustrations shown was entirely due to the stretching of the copper wire used in connecting the lever with the bellows of the player. When a similar wire was subsequently tried in my piano-player at Bangor the playing sounded exactly like that shown at the Imperial College, there being a similar slight harshness of tone and insufficient differentiation between different parts of chords. On substituting a less extensible covered steel picture-wire the softer effects again became obtainable.—*April* 24, 1913. G. H. B.]

XV. *A Graphic Method of Optical Imagery.* *By* WILLIAM R. BOWER, *B.Sc., A.R.C.S., Technical College, Huddersfield.*

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1. The object of this Paper is to show how the important properties of an optical system may be readily deduced from diagrams drawn on the following simple principle: The vertex (called the object or radiant) of a small incident pencil of rays is considered to move along a line parallel to a principal axis of the system. This line is also regarded as the direction of the incident portion of a fixed ray. Then the directions of the ray after refraction or reflection at the several surfaces do not alter as the position of the object changes. These directions are therefore the fixed loci of the images or vertices of the refracted and reflected pencils. By the diagrams obtained, the path of any ray right through the system may be traced and the successive positions of the images resulting from a progressive movement of the object exhibited. Also the constructions are such that, in the case of a thick lens, the importance of the cardinal points and the planes associated with them is demonstrated. I ask especially for the criticism of teachers, to them and their less advanced pupils the method appeals, because it visualises what is usually accomplished

by somewhat mechanical algebraic treatment. Graphic methods have previously been given by which the incident and emergent portions of a ray through an optical system can be identified. Such methods may be convenient to adopt after a time, but the beginner is like a young traveller who wants to know not only where to start and where to finish, but also something of what happens on the way.

Rule of Signs.—The one adopted is optically old-fashioned. The + sense of a line is the reverse to that of the incident light: if this travels from right to left, then the + sense of a line is from left to right. Also the + sense of vertical lines is from below upwards.

I. SPHERICAL SURFACE.

2. *Refraction at a Spherical Surface* (Fig. 1).—Let AB represent the spherical refracting surface, centre C, radius r, separating the media M_1, M_2, for which the index of refraction is μ. Assume μ=(velocity of light in M_1)/(velocity of light in M_2). Through any point O, and each of any two points B, B_1, on the circle AB, draw BO, B_1O. Draw the circular arcs Bb, B_1b_1, having their centre at O, and of radii BO, B_1O, respectively.

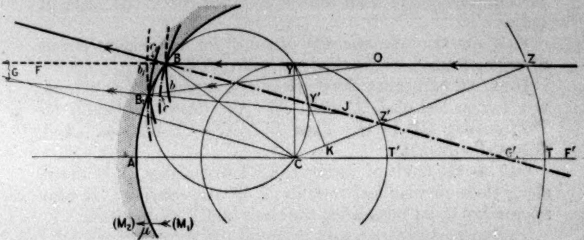

FIG. 1.—REFRACTION AT A SPHERICAL SURFACE.

Suppose Bb, B_1b_1 represent two positions of a portion of a spherical wave. This undergoes refraction between B, B_1. Let Bc, B_1c_1 be the corresponding positions of the refracted wave, and c_1BJ be the normal at c_1 to the refracted wave. Then $Bb_1=\mu \,.\, Bc_1$.

On BC as diameter describe a circle cutting BO in Y and BJ in Y′. Draw YY′, also CK perpendicular to YY′. Produce CK to cut BO in Z and BJ in Z′.

When B, B_1 are very close, the arcs BB_1, Bb, Bc are sensibly straight and perpendicular to the respective radii BC, BO, BJ. Then the triangles Bc_1b_1 and $CY'Y$ are similar. $\therefore CY=\mu . CY'$. Then Y′ is a fixed point when the line BY is fixed relatively to AB.

3. It is now convenient to deal with rays instead of waves. Let φ, φ' be the angles of incidence and refraction respectively.

(i.) Since $\mu=CY/CY'$, $\therefore \sin \varphi/\sin \varphi'=\sin CY'Y/\sin CYY' =\mu$. Thus the second law of refraction is deduced.

(ii.) The angles $CZ'B$, $CY'K$ are each equal to φ, and CZB, CYK to φ'.

Then, by similar triangles, $BZ/BZ'=CY/CY'=\mu$.

Also $\qquad BZ/CZ=BZ'/BC; \qquad \therefore CZ=\mu r$.

Also $\qquad CZ'/BZ'=BC/BZ; \qquad \therefore CZ'=r/\mu$.

Thus the locus of Z for any incident ray is the circle of radius μr and centre C. And that of Z′ is the concentric circle of radius r/μ.

Z, Z′ are the *aplanatic pair of points* for an incident pencil diverging from Z, whose chief ray is along CZ.

Draw AC parallel to the incident ray, cutting BZ′ at G′ and the circular loci of Z, Z′ at T, T′. Also draw GC parallel to BG′, cutting OB in G. Then $BG'/CG'=CY/CY'=\mu$. And $GB=CG'=BG'/\mu$.

(iii.) As the angle of incidence CBZ diminishes, Z, Z′ move along their circular loci towards T, T′ respectively; K also approaches C, its locus being the circle on CY as diameter.

4. *Refraction at a Spherical Surface: Normal Incidence.*

(iv.) At normal incidence, that is, when φ is very small, Y′ sensibly lies on CY and K coincides with C. Also Z and Z′ are close to T, T′, respectively; and, sensibly, $BZ=AT=(\mu+1)r$; $BZ =AT'=(\mu+1)r/\mu$.

Suppose F and F′ to be the respective positions of G, G′ when φ is 0. Then $AF'=BF'=\mu . CF'=\mu(AF'-AC) =\mu r/(\mu-1)$. And $BF=F'C=-(AF'-AC)=-r/(\mu-1)$.

F, F′ are respectively the first and second principal foci of a pencil incident normally at A.

(v.) When the incidence is normal it is convenient to increase largely the scale of length perpendicular to the principal axis in comparison with that along the axis. This is legitimate in an elementary discussion restricted to formulæ of the first order, provided that trigonometric relations are used cautiously. The difference of scale gives a distorted figure in which angles that are actually equal appear unequal. But the relative lengths of lines perpendicular to the axis will be maintained, and also that of lengths parallel to the principal axis.

(vi.) In Fig. 2, let AB represent the spherical surface and C its centre.

Draw any incident ray, BO, parallel to the principal axis, AC. Draw CY perpendicular to BO and cut off $CY'=CY/\mu$. Draw the refracted ray BY′, cutting AC in F′.

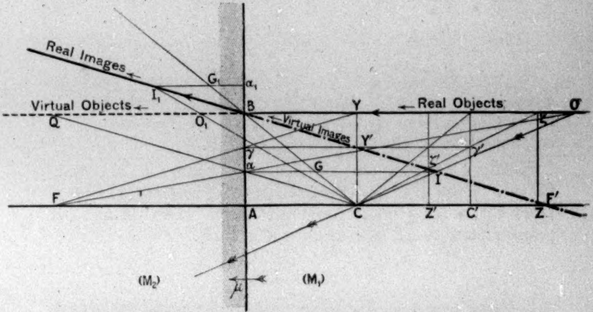

FIG. 2.—REFRACTION AT A SPHERICAL SURFACE. NORMAL INCIDENCE.

When the object is in some position O, draw CO, cutting the refracted ray at I. Then I is the image of O.

When O is at infinity, I coincides with F′. Thus, F′ is the *second principal focus*.

When O is at a point Q, such that QC is parallel to the refracted ray BF′, then the image is at infinity. Draw FQ parallel to AB, cutting AC in F. Then F is the *first principal focus*.

Then $CF'=FA$; $FC=AF'$; $AF'=AC+FA$.

Also $AF'/FA=FC/FA=CY/CY'=\mu$.

Also $FA/AC=A\gamma/Y'Y=1/(\mu-1)$.

Also $AF'/AC=AB/\gamma B=\mu/(\mu-1)$.

Through I draw aI parallel to AC, cutting AB in a. Then OaF are collinear. $\therefore$ $QO/BO=AB/aB=AF'/aI$.

$$\therefore\ (FA+BO)/BO=AF'/aI.$$

5. It is convenient to express some of these results algebraically. Write r for the radius, u, v, f, f' for the respective distances of the object, image, first and second principal foci from the refracting surface, and regard them as + when measured in the opposite sense to that of the light, and − when in the same sense. Then

(i.) $f'+\mu . f=0$. (ii.) $f+f'=r$.

(iii.) $f=-r/(\mu-1)$. (iv.) $f'=r . \mu/(\mu-1)$.

(v.) $\frac{f}{u}+\frac{f'}{v}=1$. (vi.) $\frac{\mu}{v}-\frac{1}{u}=\frac{\mu-1}{r}$.

6. *Construction for the Principal Foci*, F, F$'$, *when r and μ are given.*—Draw, as above, $AC=r$; AB; BO at any distance; CY. Make $CY'=CY/\mu$. Draw BY$'$F$'$; then γY$'$ parallel to AC, cutting AB in γ; then Yγ, cutting AC in F.

Further construction for the Aplanatic Points, Z, Z$'$.—Make $CC'=r$. Draw C$'\gamma'$ perpendicular to CC$'$, meeting γY$'$ in γ'. Draw Cγ', cutting BO in ζ and BF$'$ in ζ'. Draw ζZ and ζ'Z$'$ perpendicular to AF$'$ and meeting it in Z, Z$'$.

II. THICK LENS.

7. *Thick Lens.*—In Fig. 3, suppose two spherical surfaces to separate media M_1, M_2, M_3. Draw the principal axis through the respective centres C_1, C_2, and cutting the surfaces at A_1, A_2. Mark off F_1, F_1', the first and second principal foci of the surface through A_1, and similarly F_2, F_2' for the other surface.

Take any position, O, of the luminous point.

Draw OB_1 parallel to the principal axis. This represents the incident ray from O that is parallel to the principal axis. B_1F_1' is the corresponding refracted ray; this is incident upon the second refracting surface at B_2.

Draw C_1O. The point L where it cuts B_1F_1' is the image of O formed by the first surface. Draw F_2L cutting the surfaces in P_1, P_2 respectively. Draw P_2I parallel to the principal axis, also C_2LI meeting P_2I at I. Then I is the final image formed by the system.

Draw B_2I crossing A_1A_2 at F$'$. This line is the direction of B_2B_1 after refraction at the second surface. Then the lines

B_1O, B_1B_2, B_2I are the directions of what may be called the *second fixed ray* of the system, since their positions are unaltered, and also that of B' (the intersection of B_2F' and B_1O) for every point occupied by the object along B_1O.

Draw P_1O crossing A_1A_2 at F; produce to meet P_2I at P. Through F', B', P and F draw parallels to A_1B_1, and mark off the obvious crossing points P', A', A, B and Q.

8. In dealing with the loci of the images formed by the surfaces as the position of the object is systematically altered,

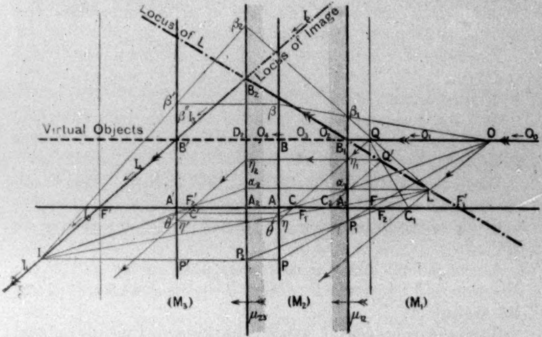
Locus of L
Locus of Image
l_2
β_2
B_2
β'
β''
l_3
β
β_1
Virtual Objects
D_2
O_4
O_3
O_2
Q
O_1
O
O_0
B'
B
B_1
Q'
L_4
η_2
η_1
α_2
α_1
A'
F_2'
A_2
A
C
C_2
A_1
F
L
F_1'
F'
C'
F_1
F_2
C_1
θ'
η'
θ
η
P_1
I
l_1
P_2
P'
P
(M_3)
(M_2)
(M_1)
μ_{23}
μ_{12}

FIG. 3.—THICK LENS.

it is convenient to consider a radiant O, that is the vertex of a normally incident pencil, to pass along OB_1 from the extreme right to the extreme left. To show that—

(i.) *F and F′ are fixed points.* All rays that emerge parallel to A_1A_2 diverge from F_2. Now, by construction, one ray P_1F through F, after refraction at the first surface passes through F_2; therefore, since the incidence is normal, all rays through F in medium M_1 are refracted so that in M_2 they diverge from F_2. F and F_2 are therefore in regard to the first surface related as object and image. Similarly, F_1' and F' in regard to the second surface. Then F and F' are fixed because F_2 and F_1' are fixed. Thus, for each position of the object the ray corresponding to P_1O passes through F.

(ii.) *The locus of P is parallel to A_1B_1.*

$$\frac{AF}{A_1F}=\frac{AP}{A_1P_1}=\frac{A_2P_2}{A_1P_1}=\frac{A_2F_2}{A_1F_2}. \quad \therefore AF=\frac{A_2F_2}{A_1F_2}\times A_1F,$$

and is constant. Then, as F is fixed for all incident rays, the locus of P is the line (and plane) through A parallel to A_1B_1.

(iii.) *The locus of B′ is parallel to A_2B_2.* Suppose a second incident ray ob_1, parallel to OB_1 (not drawn in the figure: the points corresponding to those on the diagram are indicated by small letters of the same name). Also the subsequent directions, $F_1'b_1b_2$ and $b_2b'F'$. Then

$$\frac{d_2b'}{D_2B'}=\frac{d_2b'}{A_2F'}\times\frac{A_2F'}{D_2B'}=\frac{d_2b_2}{A_2b_2}\times\frac{A_2B_2}{D_2B_2}=\frac{A_2A_1}{A_2F_1'}\times\frac{A_2F_1'}{A_2A_1}=1.$$

∴ $d_2b'=D_2B'$, and hence the locus of B′ is the straight line (and plane) through A′ parallel to A_2B_2.

The locus of P is the first principal or unit plane, that of B′ the second; A, A′ are the first and second principal points, F, F′ the first and second principal foci.

(iv.) As O approaches Q from the extreme right, the slope of OFP increases, and P and I (whose locus is B_2I) get further from A_1A_2.

When the radiant is at Q, QF is parallel to B_1A_1: then P and the image are at infinity and the emergent rays, and therefore C_2Q' (Q′ is the intersection of C_1Q and B_1F_1') are parallel to B′F′.

As the radiant leaves Q, the image passes along and in the direction B′F′, towards F′. Also P moves downwards along BA towards A.

When the image is at B′ so is P′; at the same time the radiant and P are at B.

9. *The Principal Planes are planes of unit magnification.*—(i.) B′ is the image of B and BB′ is parallel to AA′. Also, if another ray (not shown in the figure) parallel to the principal axis passes through the first and second principal planes at *b*, *b′* respectively, then *b′* will be the image of *b*. Hence an object, B*b*, will have an equal image, B′*b′*.

(ii.) To prove that any incident ray, $O\beta_1$, and its emergent portion, β_2I, intersect the first and second principal planes respectively at points, β, β', that are equidistant from the principal axis.

Since the principal planes are planes of unit magnification, an object at β on the first principal plane will have an image at, say, β'' on the second principal plane, and $A'\beta''=A\beta$. Now $O\beta_1\beta$ could be a ray incident at β_1, hence the emergent portion passes through β''. But it passes through β'. Hence β' and β'' are coincident.

Alternative proof—

$$\left.\begin{aligned}\frac{\beta P}{\beta_1 P_1}&=\frac{BO}{B_1O}=\frac{BP}{B_1P_1}\\ \frac{\beta_1 P_1}{\beta_2 P_2}&=\frac{La_1}{La_2}=\frac{B_1P_1}{B_2P_2}\\ \frac{\beta_2 P_2}{\beta' P'}&=\frac{P_2I}{P'I}=\frac{B_2P_2}{B'P'}\end{aligned}\right\}$$

Multiply the columns together.

$$\therefore\ \frac{\beta P}{\beta' P'}=\frac{BP}{B'P'}=1.$$

$$\therefore\ \beta P=\beta' P'.$$

$$\therefore\ \beta' A'=\beta A.$$

10. *The Nodal Points.*—(Fig. 3.)—Suppose the object to be at Q in the first focal plane. Draw an incident ray, QC, parallel to the fixed emergent ray, B_2F', cutting A_1A_2 in C, also A_1B_1 in η_1 and AB in η. Draw QC_1 cutting B_1F_1' in Q′. Draw $Q'\eta_1$ cutting A_2B_2 in η_2. Draw η_2C' parallel to B_2F', cutting A_1A_2 in C′, also A′B′ in η'.

$Q\eta_1$, $\eta_1\eta_2$, and $\eta_2\eta'$ are the three portions of a ray through the system. C, C′ are the *first* and *second nodal points*. Also C and C′ are related as object and image points.

11. *Properties of the Nodal Points.*—(i.) The distance between the nodal points is equal to the distance between the principal points.

Since $A'\eta'=A\eta$, the similar triangles $AC\eta$, $A'C'\eta'$ are equal. Then $A'C'=AC$ and $\therefore$ $CC'=AA'$.

(ii.) When the incident portion of a ray passes through the first nodal point, its emergent portion passes through the second, and is parallel to the incident portion.

Take any position, O, of the radiant. Draw OC cutting AB in θ. Draw $\theta\theta'$ parallel to A_1A_2. The emergent ray goes through θ'. It also goes through C′, because C′ is the image of C. It also passes through I. Thus I, θ' and C′ are collinear. Then, since $A'C'=AC$ and $A'\theta'=A\theta$, the triangles $A'C'\theta'$ and $AC\theta$ are equal. $\therefore$ C′I is parallel to CO.

12. *Construction for the Six Cardinal Points and Planes* (Fig. 4).—Draw the principal axis of the system and mark off A_1, A_2, C_1, C_2, F_1', F_2. Draw any parallel to the principal axis OB_1, cutting the surface through A_1 in B_1, and the surface through A_2 in D_2. Draw lines through B_1, F_1' and D_2, F_2 intersecting at S. Draw lines through S, C_1 and S, C_2 intersecting OB_1 at B, B′ respectively. Then B′ is the final image of B, and the object, AB, has an equal image, A′B′. Thus, B, B′ *are points on the first and second principal planes respectively.*

Let D_1 be the point of intersection of A_1B_1 and D_2F_2. Join BD_1 and produce to cut A_2A_1 in F. Also draw B_2B', cutting A_1A_2 in F′. Then F, F′ *are the first and second principal focal points.*

Draw FQ parallel to A_1B_1, meeting B_1D_2 in Q. Draw QC parallel to B_2F'. Make CC′ equal to AA′. Then C, C′ *are the first and second nodal points.*

Complete the three directions through B_1B, B_1B_2, B_2B', o the second fixed ray. For any position, O, of the object on the incident direction through B_1B, the first image, L, is on the intermediate direction through B_1 B_2, and is found at the intersection, L, of B_1B_2 with OC_1. The final image, I, is on the emergent direction through B_2B', and is at the point where LC_2 cuts B_2B'. The final images are real when I is on the actual emergent ray, and virtual when on its backward continuation.

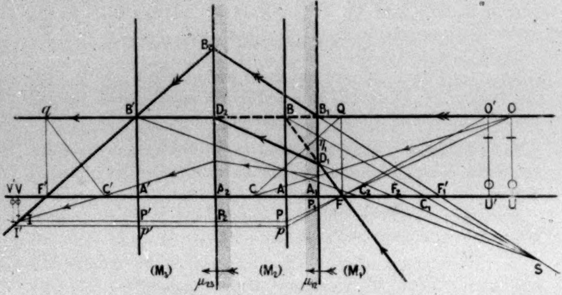

FIG. 4.—CARDINAL POINTS OF A THICK LENS.

Complete the three directions D_2B', D_2D_1, BD_1 of what may be called the *first fixed ray* through the system. The positions of images for any position of the object is found as in the previous case.

If the direction of the light is reversed, so that it is first in the medium, M_3, and incident on A_2B_2, then the positions of the cardinal points are unaltered. The names, however, must be interchanged, a first point in the former case becoming a second in this, and conversely. To demonstrate these matters in regard to the principal and focal points, consider a point object to move along B′B from the extreme left. To show that the nodal points are unaltered draw $F'q$ parallel to A_2D_2, meeting D_2B' in q. Draw qC'. Then $F'C'=F'A'—C'A'=CF—CA=AF$. $\therefore qC'$ and BF are parallel.

13. (i.) *Relations between the Focal Lengths*, A′F′ and AF (Fig. 4).—

$$\frac{AF}{BB_1}=\frac{AB}{D_1B_1}=\frac{A_2F_2}{A_2A_1}; \quad \frac{F'A'}{B'D_2}=\frac{A'B'}{D_2B_2}=\frac{A_1F_1'}{A_2A_1}; \quad \frac{BB_1}{B'D_2}=\frac{C_1F_1'}{C_2F_2}.$$

$$\therefore \frac{AF}{F'A'}=\frac{C_1F_1'}{C_2F_2}\cdot\frac{A_2F_2}{A_1F_1'}=\frac{1}{\mu_{12}\cdot\mu_{23}}=\mu_{31}.$$

$$\therefore A'F'+\mu_{13}\cdot AF=0.$$

(ii.) *Distances of the Nodal Points*, AC and A′C′.—

$$AC=FC-FA=A'F'+AF=f+f'.$$

$$A'C'=A'F'-C'F'=A'F'+AF=f+f'.$$

(iii.) When the media M_1, M_3 are the same, $\mu_{13}=1$, then $f'=-f$, and the nodal points C, C′ coincide with the principal points A, A′.

(iv.) *Standard Definitions of the Focal Lengths.*—

$$\tan FBQ=F'q/AF, \qquad \tan F'B'q=F'A'/FQ.$$

$\therefore f=F'q/\tan FBQ$, or the first principal focal length is the ratio of the length of the image formed in the second focal plane to the apparent or angular magnitude of the corresponding infinitely distant object.

Also $f'=FQ/\tan F'B'q$, or the second principal focal length is the ratio of the length of the object when in the first focal plane to the apparent or angular magnitude of the corresponding infinitely distant image.

These are the Gauss and Abbe definitions of the focal lengths of an optical system.

14. *Focal Distances, &c.*—Suppose an object, OU, and the corresponding image, VI, obtained by drawing OFP; then PP′I parallel to AA′, and meeting B′F′ in I. Then $VF'/F'A'=PA/AB=AF/FU$.

Write $AU=u$, $A'V=v$, $AF=f$, $A'F'=f'$.

$$\therefore \frac{v-f'}{f'}=\frac{f}{u-f}. \qquad \therefore \frac{f}{u}+\frac{f'}{v}=1.$$

Also $f'/f=-\mu_{13}$. $\therefore \mu_{13}/v-1/u=-1/f=\mu_{13}/f'$.

15. *Magnification.*—(i.) *Lateral Magnification* (m) or (length of image)/(length of object).

$$m=-\mathrm{IV}/\mathrm{F}'q=-\mathrm{VF}'/\mathrm{F}'\mathrm{A}'=(f'-v)/f'=f/(f-u).$$

(ii.) *Axial or Depth Magnification* (d) or (axial displacement of the image)/(very small axial displacement of the object producing it).—Let I′V′ be the position of the image when the object is at U′O′. Then VV′/OO′=VF′/QO′.

When O′ is very close to O, VV′/OO′ is the depth magnification. Also QO′=QO sensibly.

$$\therefore\ d=-(f'-v)/(f-u)=-ff'/(f-u)^2=\mu_{13}\cdot m^2.$$

(iii.) *Angular Magnification or Convergence-ratio* (w) or (angle between two emergent rays through an image point) ÷(angle between the two corresponding incident rays).

Then

$$w=\frac{\text{angle B}'\text{IP}'}{\text{angle POB}}=\frac{\mathrm{P'B'}}{\mathrm{IP'}}\bigg/\frac{\mathrm{PB}}{\mathrm{BO}}=\frac{\mathrm{BO}}{\mathrm{IP'}}=-\frac{u}{v}=\frac{f-u}{f'}=\frac{f}{f'-v},$$

$$\text{or}\quad w=\frac{\text{angle C}'\text{IF}'}{\text{angle COB}}=\frac{\text{angle C}'\text{IF}'}{\text{angle IC}'\text{F}'}=\frac{\mathrm{C'F'}}{\mathrm{IF'}}=\frac{-\mathrm{AF}}{\mathrm{VF'}}=\frac{f}{f'-v}.$$

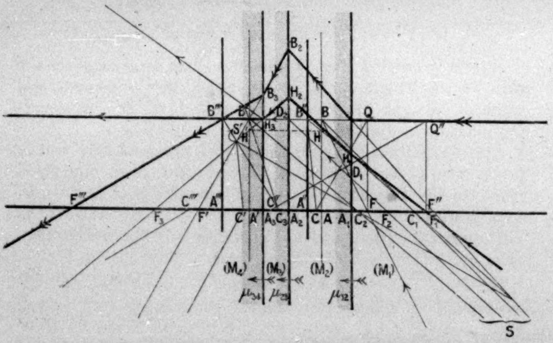

FIG. 5.—COMPOSITION OF CARDINAL POINTS.

16. *Extension to Three Coaxial Refracting Surfaces* (Fig. 5).—Suppose a third surface through A_3, centre C_3, first principal focus F_3. For the surfaces through A_1, A_2 find B, B′, F, F′, C, C′.

Let QB pierce the surface A_3 at H_3, and $H_3 F_3$ intersect $B'F'$ at S'. Draw $S'C_3$ to cut QB in B'''. Then B''' is on the *second principal plane* of the system. Let B''' B_3 intersect the principal axis at F'''. Then F''' is the *second principal focus* of the system.

Draw CB'' parallel to $S'C'$, intersecting QB in B''; then B'' is a point on the first principal plane of the system.

Let H_3F_3 pierce $A'B'$ at H'. Make $HA=H'A'$. Then F'', where $B''H$ cuts the principal axis, is the *first principal focus* of the system.

From Q'', where QB pierces the focal plane through F'', draw $Q''C''$ parallel to $F'''B_3$ and cutting the principal axis at C''. Make $C'''C''=A'''A''$. Then C'', C''' are the *first and second nodal points* of the system.

Thus the six cardinal points of a system are compounded with those of a refracting surface, and the whole reduced to six cardinal points. These may again be compounded with another refracting system and so on.

III. Spherical Surface (*continued*).

17. (i.) *Young's construction for the Refracted Ray* (Fig. 6).—From C, the centre of the spherical surface (radius r), describe circles of radii $r\mu$ and r/μ. Let the incident ray, BO, cut the former circle at Z; join CZ cutting the latter circle at Z'. Then the direction of the refracted ray is through BZ'. Z, Z' are the aplanatic pair of points for the ray ZB, $Z'B$. (*See* § 3.)

(ii.) *Alternative construction.*—On BC as diameter draw a circle cutting BO in Y. Draw the chord $CY'=CY/\mu$. Then the refracted ray passes through B, Y'.

18. *Meridian Rays.*—(iii.) YY' crosses CZ at right angles at the point K (called, by Cornu, the *junction point*).

(iv.) Let Z, Z', Z_1, Z_1' be the aplanatic pairs of points for two parallel rays incident at B, B_1, respectively. Let the corresponding refracted rays intersect at E'.

Through Z_1' draw a parallel to BZ', cutting the chord B_1B at b_2. Draw Z_1b_2e, cutting ZB in e. Now,

$$\frac{eB}{B_1Z_1}=\frac{b_2B}{B_1b_2}=\frac{Z_1'E'}{B_1Z_1'} \qquad \therefore \frac{eB}{Z_1'E'}=\frac{B_1Z_1}{B_1Z_1'}=\mu.]$$

(v.) Draw CY_1Y perpendicular to the parallels B_1Z_1, BZ; also YK, Y_1K_1 perpendicular to CZ, CZ_1 respectively. Through

K draw KF′, KF parallels to BZ, BZ′ respectively; similarly K_1F_1'. Then $FB = KF' = \mu \, . \, Z'F'$.

(vi.) When B_1 is very close to B, Y_1, whose locus is CY, is very close to Y and K_1 to K. Then sensibly F_1', and E′ coincide with F′. Also e coincides with F, for in all cases $eB = \mu \, . \, Z_1'E'$.

Also the chord BB_1 very closely coincides with the arc BB_1, and the geometrical relationships deduced from the finite chord become optical relationships for rays of light refracted at the infinitesimal arc. The above relations are therefore approximately true for a small arc.

(vii.) Suppose OO_1 is a position of a very small object moving between BZ, B_3Z_3. Join FO_1 cutting the chord and arc, B_3B, at B_4. Draw B_4I_1 parallel to BF′, cutting B_3F' at I_1. Then I_1 is the image of O_1.

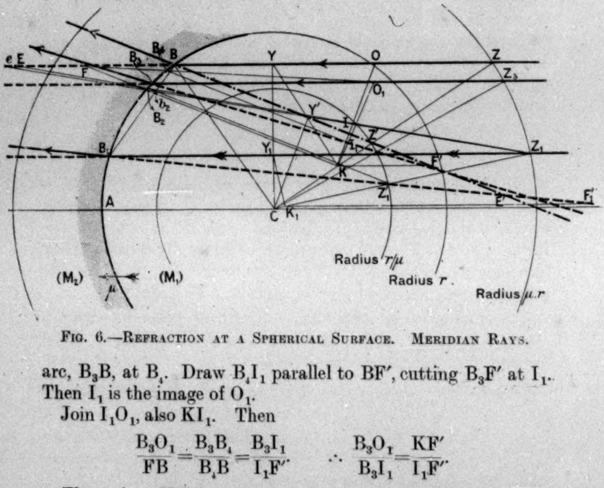

FIG. 6.—REFRACTION AT A SPHERICAL SURFACE. MERIDIAN RAYS.

Join I_1O_1, also KI_1. Then

$$\frac{B_3O_1}{FB} = \frac{B_3B_4}{B_4B} = \frac{B_3I_1}{I_1F'}. \quad \therefore \frac{B_3O_1}{B_3I_1} = \frac{KF'}{I_1F'}.$$

Then, since KF′ is parallel to B_3O_1, the points O_1, I_1, K are collinear.

When the object is very small, O_1 coincides with O, and I_1 with I. Hence the position of the image on the refracted ray BZ′, corresponding to any position O of the object on the incident ray, BZ, is at the point I where the line through OK cuts the refracted ray BZ′.

Then, considering the object to move from the extreme right to the extreme left, F, F′ are identified as the first and second *principal focal points* for narrow meridional pencils having BZ for their chief ray.

(viii.) Since (Fig. 7)

$$\frac{\mathrm{FB}}{\mathrm{BO}}=\frac{\mathrm{KF'}}{\mathrm{BO}}=\frac{\mathrm{IF'}}{\mathrm{BI}}=\frac{\mathrm{BF'}-\mathrm{BI}}{\mathrm{BI}}, \qquad \therefore \frac{\mathrm{BF}}{\mathrm{BO}}+\frac{\mathrm{BF'}}{\mathrm{BI}}=1.$$

(ix.) $$\frac{\mathrm{KF'}}{\mathrm{BY}}=\frac{\mathrm{KY'}}{\mathrm{Y'Y}}, \quad \frac{\mathrm{FK}}{\mathrm{BY'}}=\frac{\mathrm{KY}}{\mathrm{Y'Y}}, \quad \therefore \frac{\mathrm{KF'}}{\mathrm{FK}}=\frac{\mathrm{BY}}{\mathrm{BY'}}\cdot\frac{\mathrm{KY'}}{\mathrm{KY}}.$$

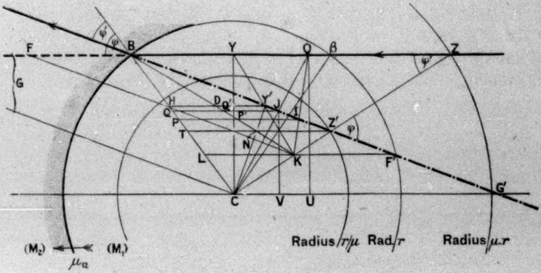

Fig. 7.—Refraction at a Spherical Surface.

Since the triangles KYC, Y′BC are similar, KY/CY = BY′/BC. Also the triangles KY′C, CYB are similar. ∴ KY′/CY′ = BY/BC.

$$\therefore \frac{\mathrm{KY'}}{\mathrm{KY}}\cdot\frac{\mathrm{CY}}{\mathrm{CY'}}=\frac{\mathrm{BY}}{\mathrm{BY'}}. \qquad \therefore \frac{\mathrm{KY'}}{\mathrm{KY}}=\frac{1}{\mu}\cdot\frac{\mathrm{BY}}{\mathrm{BY'}}.$$

Also $$\frac{\mathrm{FB}}{\mathrm{BF'}}=\frac{\mathrm{KF'}}{\mathrm{FK}}=\frac{\mathrm{BY}}{\mathrm{BY'}}\cdot\frac{\mathrm{KY'}}{\mathrm{KY}}. \qquad \therefore \frac{\mathrm{BF}}{\mathrm{BF'}}=-\frac{1}{\mu}\cdot\frac{\mathrm{BY}^2}{\mathrm{BY'}^2}.$$

(x.) $$\frac{\mathrm{KF'}}{\mathrm{BF'}}=\frac{\mathrm{NZ'}}{\mathrm{BZ'}}=\frac{\mathrm{NZ'}}{\mu\,.\,\mathrm{TZ'}} \quad \therefore \frac{\mathrm{NZ'}}{\mathrm{TZ'}}=\mu. \quad \frac{\mathrm{FB}}{\mathrm{BF'}}=\frac{\mathrm{BY}^2}{\mathrm{BY'}^2}.$$

(xi.) For BF, BF′, BO, BI, write f, f', u, v_1 respectively. And for BY put y or $r\,.\cos\varphi$, and for BY′ substitute y' or $r\,.\cos\varphi'$.

$$\frac{1}{\mathrm{KF'}}+\frac{1}{y}=\frac{1}{\mathrm{DY'}}=\frac{1}{\mathrm{HY'}}\cdot\frac{y'^2}{y^2}=\frac{\mu y'}{y^2}.$$

$$\therefore \frac{1}{\mathrm{BF}}=\frac{y-\mu\,.\,y'}{y^2}. \qquad \therefore \frac{1}{f}=\frac{\cos\varphi-\mu\,.\cos\varphi'}{r\,.\cos^2\varphi}.$$

$$\frac{1}{\mathrm{BF'}}=-\frac{1}{\mathrm{BF}}\cdot\frac{1}{\mu}\cdot\frac{y^2}{y'^2}=\frac{\mu\,.\,y'-y}{\mu\,.\,y'^2}. \qquad \therefore \frac{1}{f'}=\frac{\mu\,.\cos\varphi'-\cos\varphi}{\mu\,.\,r\,.\cos^2\varphi'}.$$

(xii.) From Fig. 7 (writing p, p' for IP, IP′ respectively), $1/p'=1/u+1/\mathrm{KF}'$.

$$\therefore \frac{y'^2}{y^2}\cdot\frac{1}{p}=\frac{1}{u}-\frac{1}{f}. \qquad \therefore \frac{y'^2}{y^2}\cdot\frac{\mu}{v_1}=\frac{1}{u}+\frac{\mu\,.\,y'-y}{y^2}.$$

$$\therefore \frac{\mu\,.\cos^2\varphi'}{v_1}-\frac{\cos^2\varphi}{u}=\frac{\mu\,.\cos\varphi'-\cos\varphi}{r}.$$

19. *Spherical Surface. Sagittal Rays.*

(xiii.) Draw (Fig. 8) CG′ parallel to BZ, cutting BZ′ in G′. Take a point B_6 on the spherical refracting surface so that a line through it drawn parallel to BZ makes an angle φ with the normal at B_6. Join B_6C, B_6G'. Then the angle CB_6G' is equal to φ', and hence B_6G' is the refracted direction of the sagittal incident ray parallel to BZ. Similarly, the refracted direction of any ray of the sagittal pencil that is incident parallel to BZ, passes through G′.

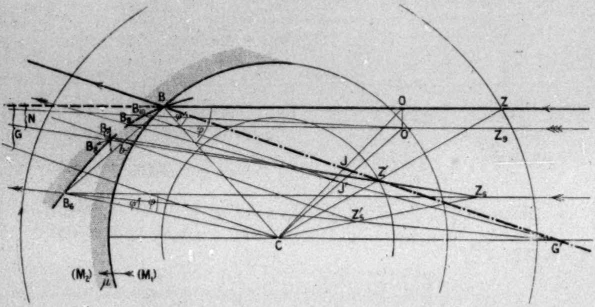

FIG. 8.—REFRACTION AT A SPHERICAL SURFACE. SAGITTAL RAYS.

For the ray B_6Z_6 let the aplanatic pair of points be Z_6, Z_6'. Through Z_6' draw $B_7b_7Z_6'$, parallel to BG′, cutting the spherical surface at B_7 and the chord B_6B at b_7. Draw b_7Z_6, cutting BZ at N and the surface at B_8. Draw GC parallel to BG′, cutting BZ at G. Then

$$\mathrm{NB}/B_6Z_6=b_7\mathrm{B}/B_6b_7=Z_6'\mathrm{G}'/B_6Z_6'=\mathrm{CG}'/B_6Z_6.$$

$$\therefore\ \mathrm{NB}=\mathrm{CG}'=\mathrm{GB},\ \text{or N coincides with G.}$$

(xiv.) When B_6 is very close to B, the points B_8, B_7, b_7 sensibly coincide. Also the chord B_6B very nearly coincides with the arc B_6B, and the geometrical relationships deduced from the finite chord become optical relationships for rays of light refracted at the infinitesimal arc. The above relations are therefore approximately true for a small arc.

(xv.) Suppose OO′ is a position of a very small object moving between BZ and B_9Z_9. Suppose these lines to be two incident sagittal rays and BG′, B_9G' the respective refracted directions. Join GO′, cutting the chord (and arc) B_9B at B_{10}. Draw $B_{10}J'$ parallel to BG′, cutting B_9G' at J′. Then J′ is the image of O′.

Join CJ′, also J′O′. Then

$$B_9O'/GB=B_9B_{10}/B_{10}B=B_9J'/J'G'.$$

Also CG′ is parallel to B_9Z_9 and equal to GB. Therefore, CJ′O′ are collinear.

Then for any position of the object along BZ the image is at the point J, where OC crosses the direction of the refracted ray BG′.

Consider the object to move from the extreme right to the extreme left. Then G, G′ are identified as the first and second principal foci for narrow sagittal pencils having BZ for their chief ray.

(xvi.) In Fig. 7, $\dfrac{GB}{BO}=\dfrac{CG'}{BO}=\dfrac{JG'}{BJ}=\dfrac{BG'-BJ}{BJ}$. Write v_2 for BJ and g, g' for BG, BG′ respectively. Therefore, $g/u+g'/v_2=1$.

(xvii.) Let BZ cut the circle of radius r at B and β.

Then $BZ=r\,(\mu\,.\cos\varphi'+\cos\varphi)$ and $\beta Z=r\,(\mu\,.\cos\varphi'-\cos\varphi)$.

Also $-g=GB=CG'=\mu\,.\,Z'G'=\mu\,.\,g'-\mu\,.\,BZ'=\mu\,.\,g'-BZ$.

$\therefore\ g+\mu\,.\,g'=BZ=r\,(\cos\varphi+\mu\,.\cos\varphi')$.

The triangles BCG′, $Z\beta C$ are similar. Therefore,

$CG'/BC=C\beta/\beta Z$. $\therefore\ -g=r^2/\beta Z=r/(\mu\,.\cos\varphi'-\cos\varphi)$.

And $BG'/BC=CZ/\beta Z$. $\therefore\ g'=\mu\,.\,r/(\mu\,.\cos\varphi'-\cos\varphi)$.

Write q for QJ. Then $1/q=-1/g+1/u$.

$\therefore\ \mu/v_2-1/u=-1/g=(\mu\,.\cos\varphi'-\cos\varphi)/r$.

20. *To find points on the Caustic formed by Refraction at a Spherical Surface.* (No figure.)—From C draw concentric circles of radii r (representing the surface), μr, $\mu r/2$, r/μ,

$r/2\mu$. (For convenience, keep one pair of compasses adjusted to radius $\mu r/2$, another pair to $r/2\mu$.)

From the radiant O draw any incident ray giving B, Z; draw CZ, giving Z′, also W, W′ where CZ cuts the circles of radii $\mu r/2$ and $r/2 . \mu$ respectively. With W as centre, radius $\mu r/2$, draw a circle cutting the incident ray BZ in Y; similarly, with W′ as centre, radius $r/2\mu$, find Y′ on the refracted ray BZ′. (Since WY=CW=WZ. ∴ CYZ is the angle in a semicircle. Similarly, CY′ is perpendicular to BZ′.) Draw YY′, obtain K. Join KO, obtain I. Join CO, obtain J.

Draw other incident rays and repeat the construction.

ABSTRACT.

The Paper contains a development of optical imagery based on elementary geometry, including limiting positions, but excluding cross-ratios, centres of perspective, &c. The method adopted is useful for teaching the properties of optical systems to those who are not essentially students of pure mathematics, and can be very satisfactorily used by those capable of draughtsmanship with mathematical instruments.

The principle of the method is as follows: The vertex (object) of a small incident pencil of rays is considered to move along a line parallel to the principal axis of the optical system. This line is also regarded as the incident portion of a fixed ray. Then the directions of the ray after reflection or refraction at the several surfaces do not alter as the position of the object changes. They are therefore the fixed loci of the images or vertices of the reflected and refracted pencils. Then the successive positions of the image resulting from a progressive movement of the object are obtained. Also the constructions are such that in the case of a thick lens the importance of the cardinal points and planes is demonstrated.

First, considering refraction at a spherical surface and assuming that μ is the relative velocity of light in the two media, a construction for the refracted portion of any incident ray is obtained from the wave principle; then the sine law of refraction and the position of the aplanatic pair of points are deduced.

When the incidence is normal it is convenient in drawing to increase largely the scale of length perpendicular to the principal axis in comparison with that along the principal axis. A distorted figure is obtained, but the relative lengths of lines perpendicular to the axis or of lines parallel to the axis will be maintained.

The principal foci for normal incidence on a single spherical refracting surface are obtained, and then the usual algebraical expressions are deduced from the figure. A convenient construction for finding the principal foci and aplanatic pair of points when μ and r are given is then shown.

For a thick lens it is most convenient to assume that the two surfaces separate three different media. After obtaining the refracted portions of a ray—called the second fixed ray—incident

parallel to the principal axis of the system, the positions of the cardinal and nodal points and planes are indicated and their more important properties deduced. A convenient construction is shown for finding the cardinal and nodal points, when r_1, r_2, μ, μ' are given. At the same time the course of the first fixed ray is determined. The first fixed ray is defined as the ray which emerges from the system in the same line as the incident portion of the second fixed ray obtained above. There is also a visualisation of the Gauss and Abbe definitions of focal length. The usual algebraic relations are deduced. After a discussion of magnification—lateral, axial and angular—it is shown that when the six cardinal points of one optical system are combined with those of another a resulting set of six cardinal points is obtained, and thus a composition of the cardinal points of various optical systems may be effected.

Returning to refraction at a single spherical surface the cases of meridian and sagittal rays are separately considered. For the former the position of Cornu's junction point is obtained in a simple manner. Finally, the usual algebraic relationships are deduced and a graphic construction for points on a caustic is given.

XVI. *Alternating-Current Magnets.* *By Prof.* E. WILSON.

READ FEBRUARY 28, 1913.

(ABSTRACT.)

IT follows from the well-known law of pull of an electromagnet that if the magnetic field alternates between positive and negative values the pull is unidirectional and intermittent. Unless means are provided to reduce the consequent chattering and vibration, the magnet is rendered useless. In the present experiments a phase-splitting device has been adopted, and consists in surrounding a portion of the pole-piece of the magnet with a short-circuited coil. The portion of the pole-piece so surrounded is sometimes said to be "shaded," and the coil referred to as a "shading" coil. The effect of this coil is to alter, not only the relative amplitudes, but the phase of the magnetic fields passing through the shaded and unshaded portions of the pole-face. The magnet used in the experiments varies the length of its gap when in action, and the influence of the gap length upon this phase displacement has been studied. When the resistance of the shading coil is such that the magnetic induction B over the whole face is substantially uniform and the gap closed, the phase displacement was 72 electrical degrees (360 deg. = 1 period). A gap length of 0·15 cm. reduces the phase-displacement to 18 deg., and consequently the minimum or "hold on" pull drops. This minimum or "hold on" pull is, of course, smaller than the average, and has to be taken into consideration in the design of the magnet. The arrangement of the shading coil above described is very effective in preventing vibration and chattering when the magnet is closed, and renders the alternating-current magnet a practical success.

With constant alternating voltage impressed upon the magnetising coils of the magnet the net pull exerted diminishes rapidly at first as the gap length increases, and tends to become more nearly constant. The R.M.S. amperes, on the other hand, steadily increase as the pull diminishes, owing to the increase in the gap length.

The observed net pull in the case of the magnet experimented upon is less than the calculated average pull, varying from 83 to 59 per cent. as the gap length varies from 0 to 1 cm.

DISCUSSION.

Mr. T. HARDING CHURTON asked at what frequencies the result had been obtained, as the chattering would, of course, be greatest at low frequencies. He was surprised to see that such a large displacement as 72 deg. could be obtained due to shading.

Prof. E. WILSON, in reply, stated that the frequency used was 50. The large displacement of phase only occurred when the air-gap was small, but it was only when the armature was actually in contact with the pole that it was required to abolish the chattering.

XVII. *The Latent Heat of Evaporation of Aqueous Salt Solutions.*

By ROBERT G. LUNNON, *B.Sc., University College, London.*

RECEIVED MARCH 5, 1913. READ MARCH 14, 1913.

THE question of the amount of heat required to evaporate 1 gramme of steam from an aqueous salt solution appears to have been dealt with experimentally by only one previous worker.* The results then obtained by Prof. Trouton were not entirely satisfactory, and at his suggestion the subject has been further investigated; the results, together with certain theoretical relations, are given in this Paper.

The cases in which the solution is a saturated one, and when it is unsaturated, require separate treatment. We shall denote by L the amount of heat, measured in calories, which is required to evaporate 1 gramme of steam from a salt solution of maximum concentration boiling under atmospheric pressure. The salt set free by the evaporation of its solvent remains undissolved at the same temperature. We shall use Q to denote the amount of heat required to dissolve a sufficient quantity of salt in 1 gramme of water, at the temperature of the boiling saturated solution, so that the resulting solution will be a saturated one.

With unsaturated solutions, evaporation by boiling is inevitably accompanied by an increase in the concentration of the solution, and if the boiling continues by a rise in its temperature. We shall therefore write $L_x dm$ for the total amount of heat required to evaporate dm grammes of steam, minus the heat accounted for by the rise in temperature of the solution; the latter is supposed to contain x grammes of salt to 1 gramme of water, and to be boiling under atmospheric pressure at t°C. This definition assumes that the heat required is proportional to the amount of steam evaporated, as long as that amount is small.

Experimental.

The principle adopted for the measurement of L involved the supply of heat electrically, at a known rate, to a boiling solution, and the measurement of the rate of formation of steam by condensing and weighing at definite intervals. Heat loss by radiation was checked by surrounding the whole calorimeter with a solution boiling at the same temperature.

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* F. T. Trouton, "Trans." Royal Irish Academy, Vol. XXXI., p. 345.

Fig. 1 is a diagram of the apparatus used. The calorimeter A was supported on glass legs inside a double-walled vessel* B, closed by a large rubber cork; and B was totally immersed in a solution contained in the large vessel C. The electric current was supplied by the leads W to an ordinary carbon filament lamp, L, enclosed in a copper jacket. This proved to be an efficient heating agent, about 400 calories per minute being developed from an 80-volt circuit. When the nitrates of sodium and potassium, which are very soluble at high temperatures, were being used, a ring stirrer, S, was added to prevent the salt caking round the lamp, and a feeding tube, F, was also fitted so that when the solution became thick more readings might be obtained, without dismantling the apparatus, by adding water and so re-dissolving the free salt. The steam was led out by the exit tube E, remaining immersed in the bath until condensed in the detachable vessel P, which was im-

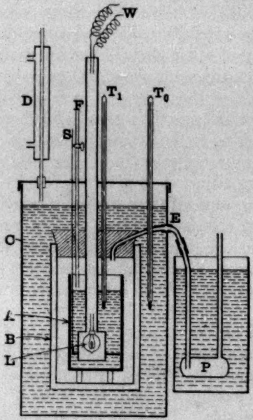

FIG. 1.

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* A vacuum-walled flask would have been ideal for the purpose of the vessel B, but repeated trials showed that such a vessel was not sufficiently strong to bear the continued heating and cooling for long.

mersed in cold water. Thermometers T_0 and T_1 indicated the temperatures of the bath and calorimeter respectively, and a condenser, D, attached to the cover of the bath, enabled the temperature of the contents of the latter to be kept constant. The electric current was measured by an ammeter, and a voltmeter was placed across the leads of the lamp. This arrangement made it necessary to multiply the current, as recorded by the ammeter, by the fraction $\frac{6058}{6058+231}$ to obtain the current passing through the lamp, since the resistances of voltmeter and lamp were 6,058 ohms and 231 ohms respectively. Both instruments were carefully calibrated twice during the work and the efficiency of other parts of the apparatus was verified.

In making an experiment, about 200 cubic cms. of a hot concentrated solution of the pure salt were placed in the calorimeter, and the bath C was then usually filled with a solution of the same salt, boiling at the same temperature. In some cases, however, an unsaturated solution of some other kind was used, its strength being adjusted to give the required boiling point. Steam, of course, came off from the inner solution at this same temperature, and this fact ensured that no water was carried over with it into the condenser. The great difficulty of wet vapour, which, as A. W. Smith* has recently proved, has disturbed much previous work on latent heats, was thus automatically avoided.

The only serious uncertainty arose from the difficulty of keeping the temperatures of calorimeter and bath exactly the same. They sometimes differed by as much as 0·5°C., and the amount of heat passing through the walls of the calorimeter was then considerable. The difficulty was much more serious when unsaturated solutions were being used, for then the constant evaporation involved a continual rise in the temperature of the solution, often at the rate of 1 deg. in 20 minutes, and the temperature of the bath could not always be raised at the same rate. It was met by determining exactly, by two different methods, the heat lost or gained, when the two temperatures differed by observed amounts. In the one method, pure water was placed in the calorimeter, and a dilute solution outside, and the consequent slightly higher temperature of the bath caused to evaporate a certain amount of steam, which was measured in the usual way. In the second method, castor oil, undergoing no appreciable change when heated for a short time, was placed inside, and was kept at a temperature slightly higher than that of the bath by a small supply of current. The heat leak was thus measured by the heat so generated.

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* A. W. Smith, Phys. Rev., 33 (1911), p. 173.

Experiments at various times by these methods gave fairly consistent results, which showed that for differences of ±2°C. the heat leak was proportional to the temperature difference, its magnitude being 9 to 10 calories per degree per minute. The irregularity due to this cause, if uncorrected, might amount to ½ per cent., and when allowance according to scale had been made it was much less. Greater accuracy than this, however, cannot be claimed for the results as a whole. The boiling of a concentrated solution is not a simple phenomenon, especially when it contains three or four parts by weight of salt to one of water; and the variation of uncertain conditions made it no easy matter to obtain consistency in the results. Experiments with pure water were made with the apparatus, and the results were very good, giving values of L between 539 and 541.

A set of readings which were obtained with a saturated solution of KCl will illustrate the method. The first column gives the time, the last two the weights of the condenser and of the steam evolved per minute.

Jan. 5th, 1912. Current: 0·245 amperes. Voltage: 79·9 volts.

t	T_1	T_0	W	w
h. m. s.				
12 19 19½	108·5°C.	109·4°C.	115·409 g.	...
0 26 19½	108·6	109·2	119·376	0·555 g.
0 38 3½	,,	109·1	...	...
0 46 3½	,,	,,	123·857	0·558
0 48 12	,,	,,	...	...
1 0 12	108·65	,,	130·510	0·554
3 43 39½	108·6	108·8	108·763	...
0 43 39½	,,	,,	114·357	0·559
0 45 35	,,	,,	...	...
0 55 35	,,	,,	119·935	0·558

$$\text{Heat supply} = \frac{EC}{J} = 14{\cdot}31 \times 79{\cdot}9 \times 0{\cdot}245 \times \frac{6058}{6289} = 270 \text{ calories per minute.}$$

Heat leak from bath = 4 calories.

$$\therefore \text{Mean } L = \frac{274}{0{\cdot}557} = 492 \text{ calories per gramme.}$$

The final means of large numbers of experiments on solutions of six common salts are given in the following Table, together with the boiling points (T) of the solutions, their strengths (x) in grammes per gramme of water, the latent heat of pure water at the temperature T (L), the heat of solution as given by the difference L_T—L, and the molecular weights (M) of the salts. The Table is arranged in descending order of maximum boiling points.

Salt.	T	L	L_T	Q $= L_T -$ L	x	M
$NaNO_3$	121·0°C.	**459**	525	66	2·18	85
KNO_3	116·8	**421**	528	107	3·38	101
NaCl	110·0	**508**	533	25	0·40	58
KCl	109·0	**493**	533	40	0·59	74
K_2CrO_4	106·8	**505**	535	30	0·82	194
$K_2Cr_2O_8$	104·8	**489**	537	48	1·03	294

These results are illustrated in Fig. 2, in which the heat of

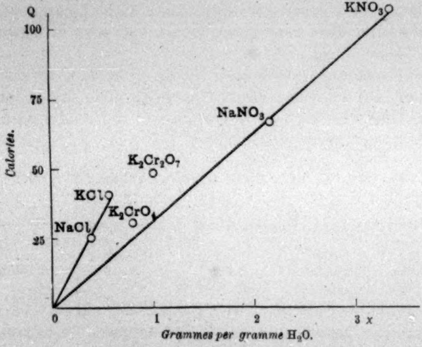

FIG. 2.

solution Q is plotted against the saturation concentration x. The most obvious connection is that between the points corresponding to salts of the same acid. The chlorides and the nitrates show definitely similar relationships, and it seems probable that for salts of the same acid Q is proportional to x, as is indicated by the two lines drawn on the diagram. The present results are not sufficient in number to test this suggestion, and, moreover, they can only be considered to be correct to ±2 calories.

Some relation might be expected to exist between the heat of solution and the strength of the solution in gramme-molecules. Fig. 3 gives the variation of Q with x/M, and it is less regular than the previous figure. This is not surprising, since the degree of dissociation of the salts would be as important as the number of molecules in solution, and this would be different for the different solutions. Unfortunately, no data exist for the dissociation coefficient, if there be such, of strong solutions.

Experiments with *unsaturated solutions* were made with two salts only, KNO_3 and $NaNO_3$. In deducing the value of L_x from the experimental observations we use the relation

$$JCE+h=mL_x+WdT,$$

where h is the heat obtained by leakage from the outside bath, m is the weight of steam formed per minute, W the water equivalent of the calorimeter and its contents, and dT the rise in temperature consequent upon the increase in concentration. This is in accordance with the definition of L_x which we have adopted, since m is always very small compared with the total mass of solution present. W is a number of considerable uncertainty, as there are no data available for the specific heat of strong solutions at high temperatures; but its value in each case was deduced in an obvious manner from special observations of the rate of rise of temperature before boiling began. Moreover, it has the small quantity dT as its factor in the equation, and an approximate value is, therefore, sufficient.

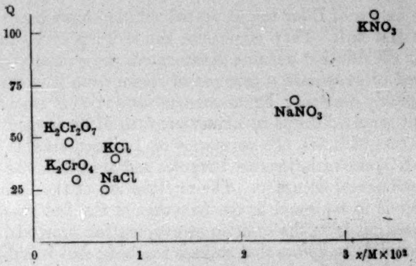

FIG. 3.

The measurements were made in the same way as for saturated solutions. The calculated values of L_x are given in the following Table for solutions of different strengths and boiling points. The numbers given for x and T are of course the mean values of these quantities over the range covered by the boiling solution during the experiment, which, as regards temperature, was usually of the order of half a degree.

$NaNO_3$			KNO_3		
T	x	L_x	T	x	L_x
100°C.	0	540	100°C.	0	540
106	0·58	553	103	0·47	541
108	0·78	550	106	1·01	541
110	0·99	554	110	1·88	543
112	1·21	547	113	2·74	540
115	1·56	545	116	3·20	543
121	2·18	459	116·8	3·38	421

The values of L for the saturated solution have been added in the last line. They emphasise the striking result that, as long as the solution remains unsaturated, the quantity of heat required to evaporate a gramme of steam from it is very approximately *constant*; but as soon as saturation is reached the amount required drops by about one-fifth of its former value. In the case of KNO_3 the constancy of L_x is remarkable, while for $NaNO_3$ the variations are irregular and are within the limits of experimental accuracy. The explanation of the final drop in value is to be found in the fact that in the last stage salt separates out from the solution as evaporation proceeds, while it does not do so before that stage is reached; but it is difficult to give a reason for the constancy of L_x while x increases, in the case of KNO_3, from 0 to 3·2.

Theoretical.

The quantity L which we have measured can be connected with other thermal constants by considering the two following thermodynamic cycles.* In each we commence with a quantity of saturated solution containing 1 gramme of water at its boiling point t°C. under atmospheric pressure p; and in the first cycle the processes are isothermal throughout.

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* The diagrams accompanying the cycles do not completely represent them, since the changes are such as to involve different parts of the working substance, being at different temperatures from one another during some of the steps; and when this occurs one point cannot completely represent the system.

First Cycle.

1. AB. Evaporate 1 gramme of steam from the solution. The heat required is L; and if w be the decrease in volume of the solution and solid salt and v the consequent volume of steam, the work done by the substance is $p(v-w)$.

2. BC. Separate the steam and compress it isothermally until the saturation pressure p_1 is reached. The work done is $\int_p^{p_1} pdv$, and the heat required is approximately $J\int_p^{p_1} pdv$. This, of course, is assuming that steam behaves as a perfect gas, but the error is actually less than $\frac{1}{2}$ per cent.

3. CD. Condense the steam. The heat required is $-L_t$ (the latent heat of evaporation of water at t°C.), and the work done is $p_1(u_1-v_1)$, where u_1 is the volume of 1 gramme of steam at pressure p_1 and temperature t.

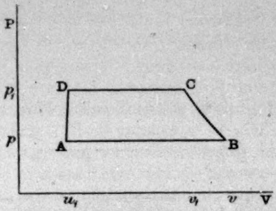

DIAGRAM 1.

4. DA. Dissolve the salt set free in the first stage, in the water obtained in the third stage. The heat required is Q, while the work done depends upon the change of volume on solution, and we shall obtain an approximate result by neglecting it.

Equating now the heat required and the work done, we have

$$L+Q-L_t=\frac{1}{J}[(pv-p_1v_1)+p_1u_1-pw].$$

We have already assumed that $pv=p_1v_1$, and the quantity p_1u_1-pw is equivalent to about 0·02 calorie, p_1 being approximately two atmospheres. We may, therefore, neglect the right-hand side of the equation, and obtain the approximate result

$$L=L_t-Q. \qquad (1)$$

Second Cycle.

1. AB. Evaporate 1 gramme of steam from the solution. This requires a quantity of heat L, and the work done is $p(v-w)$ as before.

2. BC. Cool the steam at constant pressure to 100°C., *i.e.*, until it is saturated, and let its volume then be v_0. The heat required is $-\int_{100}^{t}\sigma dt$, where σ is the specific heat of steam at constant pressure, and the work done is $p(v_0-v)$.

3. CD. Condense the steam at 100°C. If u_0 denote the volume of 1 gramme of water at 100°C. the work done is $p(u_0-v_0)$ and the heat required is $-L_{100}$, the latent heat of water at 100°C.

4. DE. Heat the water to the original temperature t, increasing the pressure so that equilibrium always exists. Let the final pressure be p_1 and the volume u_1, and write s for the specific heat of water. Then the work done is $\int_{u_0}^{u_1}pdu$, and the heat required is $\int_{100}^{t}s\,dt$.

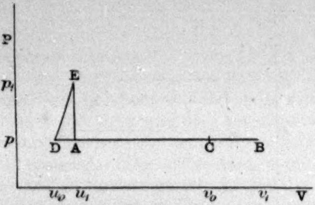

DIAGRAM 2.

5. EA. Finally, re-dissolve the salt in the water, so that the whole is in the same condition as at first. The heat required is Q, and, as in the first case, we shall obtain an approximate result by neglecting the work done.

Adding the various items and simplifying we obtain

$$L+Q-L_{100}+\int_{100}^{t}(s-\sigma)dt=\frac{1}{J}\left[p(u_0-w)+\int_{u_0}^{u_1}pdu\right].$$

The value of the terms on the right-hand side is a small fraction of a calorie in all practical cases, and for our present purpose may be omitted. If, further, we take a mean specific heat over the range from 100°C. to t°C., defined by $s_m(t-100)=\int_{100}^{t} s\,dt$, and similarly for σ, we obtain the result

$$L=L_{100}-Q+(\sigma_m-s_m)(t-100). \quad . \quad . \quad . \quad (2)$$

Another relation is obtained if we make the second cycle a reversible one, so that the Second Law can be applied. Reversibility will be ensured if the solution, in the fifth stage, be made to take place through a semi-permeable membrane ; and we shall need to define a new heat of solution, Q′, which is required under these conditions. The expression of the fact that the total change in entropy is zero is now easily found to be

$$L=\frac{T}{T_0}L_{100}-Q'+(\sigma'_m-s'_m)T\log\frac{T}{T_0}, \quad . \quad . \quad . \quad (3)$$

where T, T_0 are the temperatures t°C., 100°C. on the absolute scale, and σ'_m, like s'_m, is a new mean specific heat, defined by $\sigma'_m \log\frac{T}{T_0}=\int_{T_0}^{T}\frac{\sigma}{T}dT$.

The connection between Q and Q′ is easily found in terms of the osmotic work. The difference, Q′−Q, is equivalent to the work done by the substance, first, during the compression of the salt to the pressure P, the osmotic pressure of the saturated solution, and, second, during the movement of the osmotic membrane. If the decrease in the volume of the salt, when the pressure P is applied, be denoted by d, the first part of the work is $-Pd$. If w is the increase in the combined volume of salt and solution when 1 gramme of water is added, the osmotic work done is practically $+Pw$, for the liquid *behind* the membrane is always saturated, and therefore exerts the pressure P throughout the process of solution. We have then the approximate relation

$$Q'-Q=\frac{1}{J}P.(w-d). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

A result of some interest follows at once from equations (1) to (4), viz. :—

$$P(w-d)=J\left\{L_{100}\left(\frac{T}{T_0}-1\right)+(\sigma'_m-s'_m)T\log\frac{T}{T_0} -(\sigma_m-s_m)(T-T_0)\right\}. \quad . \quad (5)$$

In the case of an incompressible salt, the left-hand side of (5) represents the osmotic work for the solution. On the right-hand side, the only term which relates to the solution is T, the maximum boiling point; the specific heats depend only upon T also. If, therefore, the solutions of two salts have the same maximum boiling point at the same pressure, the osmotic works for the two solutions are approximately equal, and the osmotic pressures are of the same order, independently of the nature of the salts and of the concentration of their saturated solutions. Pairs of salts for which this is true are by no means uncommon, examples being the nitrate and tartrate of potassium, both boiling at 105·0°C., and potassium carbonate and zinc sulphate, both boiling at 108·0°C. We can, moveover, find the value of the osmotic pressure for a solution whose maximum boiling point T is known by substituting their appropriate values for the terms in (5). Thus, for the case of sodium nitrate, boiling at 121°C., the values of s and σ can be calculated from Tables, and the result obtained for the osmotic work is $1{\cdot}68\times 10^9$. The value of w is not much different from unity, and the osmotic pressure is, therefore, of the order of $1{\cdot}68 \times 10^9$ ergs, *i.e.*, about 1,680 atmospheres.

It is not at present possible to test our experimental results with the theoretical relations (1) to (5) which we have developed. Equations (1) and (2) involve the quantity L+Q, and its elimination leads to the approximate result for pure water

$$L_t = L_{100} + (\sigma_m - s_m)(t - 100).$$

This is known to be true for moderate values of t, such as we have assumed in making approximations. Absence of other thermal data hinders the application of these equations to the experimental results. One measurement of Q is, however, available for comparison, this having been obtained by Mr. A. W. Anscombe, B.Sc., who has commenced an investigation of the subject at this college. In his experiments, water at a high temperature and pressure is allowed to pass into a vessel containing salt at the normal boiling point (t°C.) of the saturated solution, and the initial temperature of the water is adjusted so that the final temperature of the solution formed is exactly t°C. The value of Q so obtained for $NaNO_3$ is 64·5 calories per gramme of water; and this is in good agreement with the value 66 obtained from the measurement of L with the result of equation (1).

For unsaturated solutions, theoretical results can be obtained connecting the quantity L_x with the partial heat of solution and with the heat of dilution ; but for these quantities no experimental data are as yet available and theoretical results, such as those of Kirchhoff and Clapeyron, cannot be appealed to for solutions of such large concentrations as are here mainly dealt with.

Finally, I would here record my thanks to Prof. Trouton for suggesting this subject to me, and for encouragement during the work, and also to Mr. A. W. Porter, and to Mr. Anscombe for much helpful criticism.

ABSTRACT.

The Paper records the results of a research into the latent heat of evaporation of steam from salt solutions.

The experimental method was to supply a measurable quantity of heat electrically, through a small lamp to the solution boiling inside a calorimeter. The latter was placed within a double-walled vessel surrounded by a solution boiling at the same temperature ; and the steam from the inner vessel passed out through a tube into a detachable condenser, which was weighed at intervals.

Measurements were made with the solutions of six different salts. Theoretical considerations show that the difference between the measured heat L, and l_T the known heat of evaporation of water at the same temperature, is the heat of solution Q ; and the present results indicate that for salts of the same acid Q is proportional to the concentration.

For unsaturated solutions of KCl and NaCl the interesting result is found that the heat of evaporation is approximately constant for all concentrations until saturation is reached.

DISCUSSION.

Mr. F. E. Smith remarked that in a similar apparatus he had used for the distillation of water he found that priming did take place. It would have been more satisfactory if the author had used a solution of NaCl and tested the distillate for its presence.

The Author stated that he had tested the distillate in one case, but not with NaCl.

XVIII. *On Errors in Magnetic Testing due to Elastic Strain.* By ALBERT CAMPBELL, *B.A.*, *and* H. C. BOOTH, *A.R.C.Sc.* (*From the National Physical Laboratory.*)

RECEIVED FEBRUARY 14, 1913. READ APRIL 11, 1913.

SOME years ago, in the course of magnetic tests on sheet material, Mr. T. L. Eckersley and one of the authors noticed that considerable errors were sometimes introduced by even slight bending of the sheets or strips. Mr. Eckersley worked out the stresses, and these, in combination with the known effects of compression and tension,* were found to be sufficient to account in a general way for the observed results. Only a few systematic experiments were made at the time, but recently a more comprehensive series have been carried out. The following description of some of these will serve the object of the present Paper, which does not aim at absolute results, but rather indicates the order of the errors that may occur in various cases. Hence the reader must not be too critical of the methods employed, as their accuracy was sufficient for the purpose in view. Two methods, A and B, were used.

In method A a single length of strip, which was flat in the unstrained condition, was bent into ring form, care being taken to avoid permanent set as much as possible. The slightly overlapping ends were securely clamped, and the ring was evenly wound with secondary and primary coils. These windings were of thin wire arranged close to the strip so as to keep the ring as flexible as possible. While in this condition of temporary strain, the ring was tested for permeability and hysteresis by the ballistic method. The greater part of the temporary strain was then annulled by changing the circular form into a square by sharp bends at four places. The windings were not removed for this alteration. The ballistic tests were then repeated. As will be seen by the results given below, the changes observed were considerable. The effect of the bends at the corners (which would worsen the magnetic qualities) has been ignored, and the square form has been considered as giving the condition of " no strain."

In method B two strips of the material, with suitable uniform windings, were clamped flatwise at the ends by two solid yokes, forming with them a long rectangular magnetic circuit. This circuit could be tested with the strips flat or bent to any desired amount. Permanent set at various curvatures could be applièd, and the strips could be restored to their original flat form after any of the bending processes.

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* *See* Ewing " Magnetic Induction in Iron and Other Metals," Chapter IX. (Third Edition, 1900).

Tests were made both on ordinary (*unalloyed*) soft transformer sheet and on silicon iron (containing about 3 per cent. of silicon). The latter is much the harder material, and showed correspondingly greater alterations due to temporary strain.

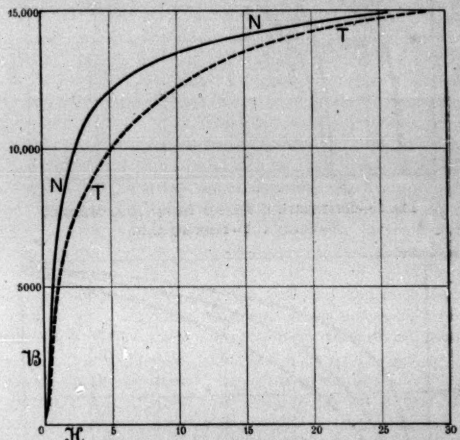

FIG. 1.—PERMEABILITY OF SILICON IRON. RING METHOD.
N = No strain. T = Temporary.

The accompanying curves and tables show some of the results obtained. The curves are marked in each case as follows :—

N, No strain.
T, Temporary strain.
P, Equal permanent strain.

Tables 1, 1A, 2 and 2A (with Figs. 1 and 2) refer to the ring method, and the others to the yoke method. In the latter method, after the effects of temporary strain had been tested, the strips were allowed to unbend back to the straight condition. When re-tested they gave practically the same results, both for permeability and hysteresis, as in the original unstrained condition. This interesting fact is not indicated in in Fig. 3. It will be seen from that figure that the temporary

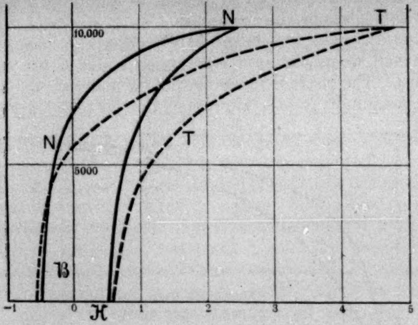

FIG. 2.—HYSTERESIS IN SILICON IRON; RING METHOD.
N=No strain. T=Temporary strain.

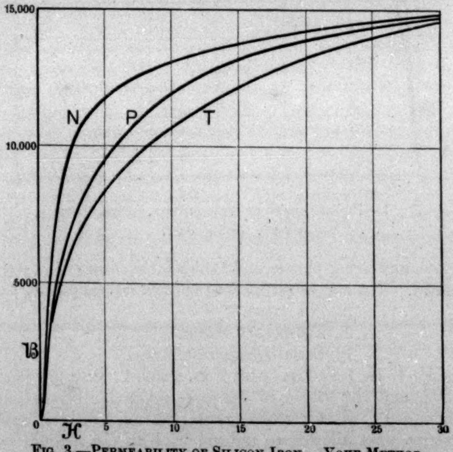

FIG. 3.—PERMEABILITY OF SILICON IRON. YOKE METHOD.
N=No strain. T=Temporary strain. P=Equal permanent strain.

strain has considerably greater effect on the permeability and the hysteresis than the equal permanent strain has.

TABLE 1.

Ordinary transformer iron (0·31 mm. thick). T, ring 23 cm. diameter; N, bent into square.

$\mathcal{H}$	N, no strain.		T, temporary strain.		μ_0/μ_1
	$\mathfrak{B}$	μ_0	$\mathfrak{B}$	μ_1	
1	1,800	1,800	1,390	1,390	$1{\cdot}29_5$
2	5,780	2,890	4,600	2,300	1·26
5	10,080	2,016	8,980	1,796	1·12
10	12,340	1,234	11,780	1,178	1·05
20	13,880	694	13,780	689	1·01
50	15,530	311	154,80	310	1·00

TABLE 1A.

Same ring as in Table 1.

$\mathfrak{B}_{\max.}$	Hysteresis loss. Ergs per c.c. per cycle.		Increase per cent.
	N, no strain.	T, temporary strain.	
5,000	1,117	1,214	8·7
10,000	3,628	3,968	9·4

TABLE 2.

Silicon-iron (0·31 mm. thick). T, ring 50 cm. diameter; N, bent into square.

$\mathcal{H}$	N, no strain.		T, temporary strain.		μ_0/μ
	$\mathfrak{B}$	μ_0	$\mathfrak{B}$	μ	
1	6,510	6,510	4,640	4,640	1·40
2	9,330	4,665	6,980	3,490	1·34
5	12,090	2,418	9,980	1,996	1·21
10	13,440	1,344	12,230	1,223	1·09
15	14,110	941	13,510	900	$1{\cdot}14_5$
25	14,870	595	14,700	588	1·01

TABLE 2A.

Same ring as in Table 2.

$\mathfrak{B}_{\max.}$	Hysteresis loss. Ergs per c.c. per cycle.		Increase per cent.
	N, no strain.	T, temporary strain.	
5,000	468	558	19·2
10,000	1,552	1,850	19·2

TABLE 3.

Silicon-iron (0·42 mm. thick) by yoke method.
N, strips straight.
T, bent temporarily to arc of circle 18 cm. radius.
P, bent with permanent set to arc of same circle.

$\mathcal{H}$	N, no strain.		T, temporary strain.		P, permanent set.	
	$\mathfrak{B}$	μ	$\mathfrak{B}$	μ	$\mathfrak{B}$	μ
1	6,580	6,580	4,220	4,220	4,700	4,700
2	9,210	4,605	5,710	2,855	6,580	3,290
5	11,770	2,354	8,340	1,668	9,450	1,890
10	13,100	1,310	10,640	1,064	11,900	1,190
20	14,170	708	13,200	660	13,870	693
30	14,820	494	14,550	485	14,740	491

TABLE 3A.

Same sample as in Table 3.

$\mathfrak{B}_{\text{max}}$	Hysteresis loss. Ergs per c.c. per cycle.		
	N, no strain.	T, temporary strain.	P, permanent set.
10,000	1,458	1,834	1,754

It should be mentioned that the results here given are only examples of cases in which the effects were somewhat pronounced; in some other samples which were tried the same amount of bending gave much less change. This is quite to be expected, for the total effect is a differential one, the permeability being sometimes increased by the tension and decreased by the compression resulting from the bending. It is, however, to the existence of the worse cases that attention has to be drawn.

In the Richter method of testing hysteresis loss the sheets are bent into the form of a ring of 50 cm. diameter. Although this gives a curvature which appears to be very moderate, the results given in Table 2A show that the slight bending may cause errors (in the hysteresis loss) of more than 10 per cent. From the above investigation the general conclusion follows that in all magnetic tests it is important that the greatest care should be taken to avoid not only permanent deformation but also temporary elastic strain.

ABSTRACT.

In magnetic tests on sheet material considerable errors may occur if the sheets or strips are tested while in bent form. These errors, which are in general agreement with the known effects of compres-

sion and tension, were investigated experimentally with one or two forms of magnetic circuit similar to those sometimes occurring in practice. In one method a single length of the strip was bent into ring form with ends clamped together. This was wound with flexible primary and secondary coils, and tested for permeability and hysteresis, while in the condition of temporary strain. The temporary strain was then annulled by changing the circular form into a square by sharp bends at four places. The magnetic tests were repeated, and usually a considerable alteration was observed. For example, a silicon-iron ring 0·3 mm. thick and 50 cm. in diameter (the size used in Richter's method of testing hysteresis and eddy current losses) showed a decrease of 40 per cent. in the permeability for $H=1$ due to the bending. The hysteresis loss was increased by 19 per cent. In another method the ends of two strips were clamped in yokes, and tests were made with different amounts of bending. It was found that temporary strain has considerably greater effect on the permeability than equal permanent strain has.

DISCUSSION.

Prof. C. H. **Lees** pointed out that the authors had not clearly distinguished between the effects of stress and strain, and that the results would be more intelligible if this distinction were made. The condition called by them "temporary strain" was strain plus stress, while that called "equal permanent strain" was the strain without the stress. From the point of view of the molecular dynamical theory he thought the effects of stress ought to be separated from those of strain.

Dr. S. W. J. **Smith** expressed interest in the results and, referring to the difference between the effects produced by temporary stress and by permanent strain, thought there was no reason to expect any simple relation between them. Under different fields, in the same material, or under similar fields, in other materials, the effects might even be opposite to one another in sign.

Prof. T. **Mather** asked if the sheets were ever annealed after being bent, before testing them by Richter's method, as this would seem to remove the objections against the method.

Mr. A. **Campbell**, in reply, admitted the advisability of separating the effects of stress from those of strain. The advantage claimed for Richter's method was that it in no way mutilated the sheets. If they were annealed after they were bent into a cylinder this advantage would be destroyed.

XIX. *Note on Cathodic Sputtering.* *By* G. W. C. KAYE, *B.A., D.Sc. The National Physical Laboratory.*

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As is well known, if a current is sent through a discharge tube at low pressures, the glass adjacent to the cathode often becomes coated with a deposit of metal to an extent which depends on, among other things, the material of the cathode and the nature of the gas in the tube. The anode, on the contrary, shows little or no such effect.

This cathodic "sputtering" was noticed in the very early days of vacuum tubes: Plücker (1858) and Geissler remarked on it. Dr. Wright, of Yale, in 1877 used the method to platinise glass. The subject has been investigated by a number of workers, including Sir Wm. Crookes in 1891 ("Proc." Roy. Soc. A., 50, p. 88), Holborn and Austin, in 1903, at the Reichsanstalt ("Phil. Mag.," 8, p. 145), Houllevigue ("Ann." Chim. Phys., 1909-10). and Kohlschütter and his coadjutors, whose work is described in a recent series of Papers in the "Zeitschrift für Elektrochemie" (1906-1912).

The sputtered metal is shot off as small aggregates of molecules, apparently of the same order of magnitude as the particles in colloidal solutions. They appear to be projected approximately normally from the cathode, and to carry a negative charge, though, on account of their relatively large mass, the streams of metal are not deflected by a magnet to anything like the same extent as cathode rays. It does not appear that the disintegration of the cathode plays any appreciable part in the transmission of the current.

Cathodic sputtering has proved very useful as a means of preparing films and mirrors, especially in the case of metals intractable by the usual methods. It is found that the deposits only settle on surfaces which are positive with respect to the cathode. In practice this is secured by joining the surface to be plated to the anode.

The exact mechanism of the production of the sputtered particles is doubtful and has been the subject of a recent controversy. It is definitely established, however, that the amount of sputtering depends on :—

1. *The Nature of the Cathode.*

It is a rule, to which there is a number of exceptions, that the amount of sputtering is roughly proportional to the chemical equivalent of the metal. To particularise, the effect in air with aluminium and iron is small. With palladium, platinum, gold, silver, copper, cadmium and tin sputtering is usually marked.

2. *The Temperature of the Cathode.*

If the temperature of the cathode be raised appreciably either by extraneous means or by the discharge itself, metals of high volatility, such as cadmium and zinc, have their sputtering properties greatly enhanced.

3. *The Nature of the Gas.*

Hydrogen, nitrogen and carbon dioxide are unfavourable to the cathodic disintegration of most metals. Oxygen and more especially the monatomic gases, helium, neon, argon, krypton, xenon and mercury vapour bring about marked sputtering of nearly all metals; argon is particularly efficacious in this respect. The nature of the gas furthermore controls the appearance and properties of the deposit.

4. *The Current Density in the Discharge Tube.*

The disintegration appears to be roughly proportional to the square of the current density, so that in sputtering experiments it is beneficial to use small (wire) electrodes and as large an induction coil as may be expedient.

5. *The Fall of Potential at the Cathode.*

The volatilisation of the cathode is augmented by increasing the potential on the tube and so increasing the potential drop at the cathode, and this is most readily controlled by regulating the pressure of the gas. Sputtering is much more pronounced at low pressures than at high, though it is possible to force the pressure too low.

An increase in the potential drop at the cathode also increases the range of projection of the particles.

It appears to be essential that the cathode-fall shall exceed a certain minimum voltage before the metal becomes ionised and disintegrated to any appreciable extent; and this fact is sufficient to reveal, under favourable conditions, a marked difference between the sputtering which occurs at the various parts of a cathode of irregular shape. The potential gradient attains a maximum at points and edges; and accordingly it is at such places that the tendency for the metal to sputter predominates.

A discharge tube which the writer has used at the National Physical Laboratory illustrates this feature very strikingly. The electrodes were cylinders made by bending thin sheet aluminium so that two opposite edges came nearly together. These cylinders fitted very loosely within the tube, which was filled with helium—a gas somewhat favourable to the sputtering of aluminium. The pressure was not very low—about that which displays the helium spectrum to advantage—so that the tube ran easily, the dark space was small, and the potential applied was never great. As Fig. 1 indicates, there is no trace of sputtering at the anode, but an examination of the tube near the cathode shows that, while the glass facing the sides of the cylinder is untouched by the deposit, there is a brilliant black mirror on the glass immediately beyond each end of the cylinder.

Evidently the particles of metal were shot off exclusively from the region of the edges of the cylinder where the potential gradient reached a maximum. Elsewhere, the field attained was not strong enough to cause the emission of metal. This is borne out by the fact that there is a narrow band of deposit on the glass parallel to and directly facing the slit where the two edges of the bent plate come together. These two edges did not quite meet, and the intervening space is reflected in the band of the deposit which shows a streak of clear glass along the middle of its length. There is an extensive and ill-defined deposit on the inside of the aluminium tube opposite the slit. This is explained by the fact that the potential of the inside of the sheet is positive with respect to the edges.

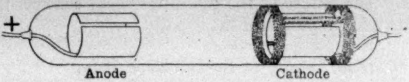

FIG. 1.

Fig. 2 is a photograph of the sputtered deposit on the glass with the cathode removed. Fig. 3 shows the outer surface of the cylindrical cathode which has been flattened out for the purposes of the photograph; the extent of the stained active region round the edges is clearly discernible. Both Figs. 2 and 3 are full size.

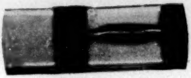

FIG. 2.

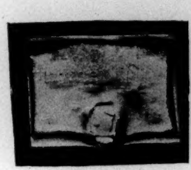

FIG. 3.

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While the present case serves as an interesting illustration of the distribution of electric density on a cylindrical conductor, it may point a moral in X-ray tube construction. The ordinary potential-excess of the edges of a cathode of an X-ray bulb is accentuated by the disposition of the cathode which, for greater steadiness and hardness of the discharge, is always set back a little into a side tube. The glass walls of this tube become negatively charged by leakage from the cathode and accordingly the cathode rays are repelled to the centre of the cathode where they leave as a sharply defined pencil along the axis. This is the case whether the cathode is concave or plane. Now, since the centre of the cathode is the main source of the cathode rays which carry the current through the tube, it follows that the potential at the centre of the cathode is less than that at the edges and thus the ordinary edge excess of potential is emphasised This is the explanation, I think, why sputtered deposit from the edges can be found in the central area of the cathode of an old X-ray bulb.

A deposit from the edges of the cathode forms also on the glass in the vicinity. This is objectionable and might possibly be prevented by constructing the cathode wholly free from sharp edges; the resulting greater uniformity of intensity might ensure that, with a soft tube at any rate, the active sputtering potential would not be reached.

Probably the point is of greater importance with the anticathode. No coil discharge is wholly free from the inverse current, and it is during this phase, when the anticathode officiates as cathode, that most of the mischievous blackening of an X-ray bulb occurs, the more especially with a metal like platinum. This might be largely got over by making the anticathode approximate in shape to a sphere.

ABSTRACT.

The Paper gives an account of the volatilisation of an aluminium cathode in a discharge tube containing helium. The sputtered deposit on the glass indicates that, under the conditions which prevailed, the disintegration was restricted to the edges of the cathode and did not occur elsewhere. Accordingly the complete outline of the cathode (made by rolling a sheet of aluminium into a nearly complete cylinder) was traced out by the deposit on the walls of the tube.

DISCUSSION.

Prof. J. W. Nicholson asked if the size of the particles bore any relation to the atomic weight of the metal.

Prof. T. **Mather** asked if Dr. Kaye had made any tests of the conductivity of the films, and whether they obeyed Ohm's Law.

Mr. D. **Owen** inquired how adherent the films were to the glass.

The **Author**, in reply, stated that the size of the particles depended chiefly upon the nature of the gas in the tube. The method had been considerably used for preparing extremely high resistances though the film had to be annealed first. He believed the thinnest films did not obey Ohm's Law accurately. The adherence of the film varied with the metal. Gold adhered to glass well, but platinum was easily rubbed off till it had been heated to redness.

XX. *On Vibration Galvanometers with Unifilar Torsional Control.* *By* ALBERT CAMPBELL, *B.A.*

RECEIVED APRIL 11, 1913. READ APRIL 11, 1913.

IN vibration galvanometers, whether of moving-circuit or moving-magnet type, the control torque employed has usually been either elastic or magnetic, or a combination of both. I am not aware that either gravity or electrostatic control has been utilised in any types. The following Table gives the outstanding features of the more familiar forms of vibration galvanometers :—

Designation.	Type.	Current circuit.	Control.	Final tuning by
1. Rubens, 1896	Moving iron ...	Polar coils......	Wire torsion and magnetic	Displacement of magnet poles
2. Wien, 1901 ...	Moving magnet	Electromagnet	Ditto	Ditto
3. Campbell, 1907	Moving circuit	Moving coil...	Elastic (bifilar)	Alteration of wire tension
4. Duddell, 1909.	Moving circuit	Moving loop...	Ditto	Ditto
5. Drysdale, 1910	Moving iron...	Fixed coil......	Magnetic	Alteration of field by magnetic shunt

In order to obtain the full sensitivity in a vibration galvanometer it is essential to be able to vary the natural period of the vibrating system very gradually so as to get very fine tuning. It will be seen from the above Table that in different types the final tuning is done in a variety of ways. In the unifilar galvanometers of Rubens and Wien, although the control is largely due to the torsion of a wire, the final adjustment is made by slight alteration of the magnetic field. Some time ago I discovered that, if a single strip (of hard phosphor bronze) is used instead of a wire, the torsional control can be varied to a large extent by merely altering the tension of the strip. In fact the strip behaves like a bifilar suspension, and the tuning can be done with great exactness by first adjusting the effective length with a sliding bridge, and then making the fine adjustment by altering the tension.

I find that this property of certain strips, which has proved so useful, has also been noticed by Prof. L. R. Wilberforce, and investigated recently by Mr. H. Pealing ("Phil. Mag." p. 418, Vol. 25, March, 1913). Mr. Pealing gives examples of strips in which the torsional rigidity was increased by 40 to

50 per cent. by heavy loading; he finds that the anomalous effect is removed by annealing, and he attributes it to overstrains during the process of manufacture.

I have used the new system of unifilar tuning both in moving-coil and moving-iron vibration galvanometers, and find it very convenient. In the moving-coil type the coil is made quite detachable, having contact hooks at top and bottom, which are caught by minute hooks on the ends of the upper and lower strips. The sensitivity of such a galvanometer depends a good deal on the size of mirror which is considered sufficient. With a mirror of 15 sq. mm. area a sensitivity at 100 ∾ per second of 50 mm. at the meter distance per microampere is obtained, the effective resistance being about 700 ohms. Very good sensitivity can be obtained (at low frequencies) with strips of only 1 cm. to 2 cm. length, and the instrument thus can be of very moderate height.

The unifilar tuning is also successful for galvanometers of the moving-iron type. The moving iron may be in various forms—for example, in small permanent magnets (like Wien's) or in a polarised thin strip, as in one of Blondel's oscillographs. Mr. Pealing observed that continued tension for some weeks gradually diminishes the anomalous effect. I have had one galvanometer under full working tension for five or six weeks, and it is still behaving very well; but further observation on this point is desirable.

ABSTRACT.

The Author exhibited a moving-coil vibration galvanometer in which a novel principle is used to obtain the fine adjustment of the control torque requisite for accurate tuning. He has found that in a phosphor-bronze strip under tension the torsional rigidity is considerably increased as the tension is raised. This anomalous behaviour of such strips has also been noticed by other observers (H. Pealing, "Phil. Mag.," March, 1913). If unifilar (strip) suspensions are used in a vibration galvanometer (whether of moving-coil or moving-iron type) the tuning can be done in just the same way as with bifilar suspension. In the moving coil instrument minute hooks on the ends of the strips engage in contact hooks at the top and bottom of the coil, which is easily detachable.

With a mirror of 15 sq. mm. area, at 100∾ per second, a sensitivity of 50 mm. at a metre per micro-ampere can be obtained, the effective resistance being about 700 ohms.

DISCUSSION.

Prof. T. Mather asked Mr. Campbell if he did not think the variation of period was possible because of the bifilar behaviour of a flat strip.

Prof. C. H. **Lees** remarked that it seemed natural to expect a strip to behave like a bifilar. He thought Pealing's dismissal of this explanation would only be valid in the case of long strips.

The **Author**, in reply, stated that to some extent the result must certainly be due to bifilar behaviour. The effect due to this cause decreased as the length of the strip increased. In his own case where the length was only about 1 cm. it was probably large, but he did not think that this could be the whole explanation of the results in Pealing's experiment where a length of 40 cm. was used.

XXI. *Interference of Röntgen Radiation* (*Preliminary Account*).

By Prof. C. G. Barkla, *F.R.S.*, *and* G. H. Martyn, *B.Sc.*

Received April 3, 1913. Read February 28, 1913.

Friedrich, Knipping and Lane recently showed that when a narrow pencil of Röntgen radiation traverses a crystal many diffracted pencils proceed in various directions around the directly transmitted pencil from the portion of crystal traversed. Assuming the molecular structure of a crystal of zinc blende to be such as to form a space grating, they concluded that the images observed on the photographic plate would be produced if the radiation contained several conspicuous homogeneous constituents—that is radiation of several distinct wave-lengths.

Mr. W. L. Bragg pointed out that the directions of the pencils of radiation proceeding from the crystal could all be accounted for by considering simple reflection to take place at the various planes within the crystal containing the largest number of molecules, and that the radiation was analogous to white light giving a continuous spectrum in Röntgen radiation between certain limits of wave-length. The theory is in essentials the same as that of Laue, but it limits the applicability of Laue's theory in one direction and extends it in another.

We were engaged in investigating the phenomenon of the transmission of radiation through crystals, when an announcement of this important modification by W. L. Bragg was made. Two short notices were shortly afterwards published by us in "Nature." In the following Paper we now give a preliminary account of some of the experiments we have made up to the time of writing.

A crystal of rock-salt, which is of the simple cubic form, was placed with one set of cleavage planes horizontal, and one of the two sets of vertical cleavage planes in the geographical meridian say, consequently, with the other set in east-west vertical planes. We will for convenience call the two sets of vertical planes the NS and the EW cleavage planes.

An X-ray tube was placed below the crystal and was arranged to move in circular grooves in a NS plane around a point in the small crystal as centre. The angle of incidence of the Röntgen radiation on the crystal could thus be varied, while the distance of the antikathode from the crystal remained constant.

Reflexion of X-Rays.—First the rays were allowed to pass through a very narrow circular aperture in a lead cap fixed to the bulb of the X-ray tube, and from this through a very narrow slit in a horizontal lead screen immediately below the crystal, the length of the slit being in a NS plane. Radiation passing through the slit was incident on the EW planes in the crystal at various angles as shown in the Fig. 1. It was found by placing photographic plates at various distances above the crystal that the principal emergent pencil (excepting that directly transmitted) was one proceeding in the direction shown in the figure, indicating reflection from the EW cleavage planes. Thus the radiation from a point source diverging in a NS plane and falling on the EW cleavage planes produced a converging pencil which passed through a point focus (the vertical thickness of the crystal being negligible in comparison

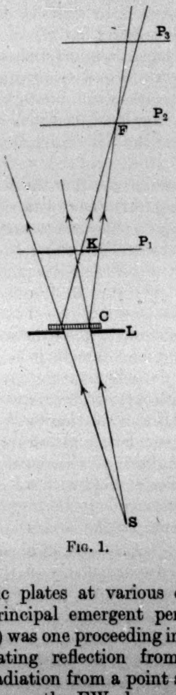

FIG. 1.

with the other distances concerned). When the slit below the crystal was widened so that it became a wide rectangular aperture, photographic plates placed above the crystal showed a continual divergence of the reflected pencil in an EW plane, but a convergence in a NS plane, until the distance of the photographic plate above the crystal was approximately equal to that of the point source below. Beyond this it diverged in a NS plane. Thus at a distance above the thin crystal equal to that of the source below, the secondary radiation was brought to a line focus, the line being horizontal in an EW direction. When the incident pencil was rotated in the NS plane (the plane of the figure), the secondary pencil rotated in the same plane, but in the opposite sense through an equal angle approximately. When the incident pencil was turned from the N side to the S side of the EW cleavage planes, the secondary pencil also moved to the S side. In these experiments the angle of incidence varied only from about 80 deg. to 90 deg.

Thus an examination of the shape, direction and variation in direction of the secondary pencil show that this was a radiation regularly reflected from the cleavage planes within the crystal.

We must state here, however, that in later experiments when a much greater length of crystal was exposed, and the X-ray tube was removed to a much greater distance, the secondary beam did not converge to a line focus in the position we had expected on the simple reflection theory. Whether this was due to irregularity in the crystal structure or to the inadequacy of a simple reflection theory we cannot at present say. A tilt of cleavage planes through an angle of about 15 minutes would produce the deviation observed. We have, however, found the laws of regular reflection to hold with a fair degree of accuracy in our experiments with small crystal fragments when the angle of incidence was between 80 deg. and 90 deg.

Fringe System.—It has been seen that using a small source of radiation, and allowing a pencil to pass through a narrow slit with its length in a NS direction in the lead plate below the crystal, different portions of this pencil were incident at different angles on the EW cleavage planes. A photographic plate in position P, was affected in the position K by the radiations reflected at various angles, a definite point on the photographic plate corresponding to a definite angle of incidence and reflection. It was found that the narrow band produced on the plate with its length in a NS direction was broken up into a series of approximately equal-spaced portions by maxima of photographic effect. At one end of the system there was complete separation of the maxima by portions of the plate which were not affected to any appreciable extent.

Such a system could be explained by interference between pencils of radiation reflected from a large number of equal-spaced EW cleavage planes in the crystal, the source being small, and the radiation producing photographic action being fairly homogeneous, that is consisting of waves of length within fairly narrow limits. This was originally suggested by us as a possible explanation which seemed adequate to the phenomena we had observed. Later experiments have not given support to this simple explanation, though the phenomena still appear to be due to interference.

On removing the lead diaphragm immediately outside the X-ray tube, and thus exposing the crystal to the whole of the antikathode, the source was, of course, more extended. The result was in some cases the almost complete disappearance of the above fringe system due to overlapping. When, however, the antikathode was situated at a considerable distance from the crystal the fringes were still quite clearly marked, as the source was again small compared with the distance from the crystal.

Up to the time of writing five crystal fragments have been used (two, however, being from the same large crystal) and these fringes have been observed with each. In some cases the fringe system became blurred owing to obvious irregularities in the crystal, and in parts it was indistinguishable.

The most extended system observed was one in which at one end of the image the fringes were very distinctly separated, while at the other they appeared to coalesce in pairs. Assuming each fringe at the latter end still to contain two constituent fringes, the full series consisted of 26 fringes.

Band System.—Using a long rectangular aperture it was seen that the rectangular image obtained by reflection as before was broken up into a system of bands separated by spaces showing little or no photographic action. These spaces coincided in position with minima of the fringe system, while each band of this second larger system usually contained eight complete fringes of the first system. (These appeared as four in the coalesced form.) That is, a long rectangular image was divided into a number of bands of approximately equal length, each band being very definitely separated from the next, and each containing a number of fringes which were usually much less clearly marked.

In addition to the fringe system and band system above described, there was a variation in intensity of the image from band to band. A sufficiently large bumber of bands has not been observed to enable us to see if this variation in intensity is regularly periodic, that is if there is a larger system of bands, containing the system described above, Not more than six bands have been observed in one image.

Brushes.—Other pencils of radiation not obeying the laws of regular reflection have been found to be emitted by some crystals. Those are shown on the photographic plate by brushes spreading out from the image due to the directly transmitted radiation. These are thus due to pencils of radiation apparently spreading out in planes passing through the direction of propagation of the primary radiation.

Though sufficient experiments have not been performed to enable us to arrive at a final conclusion as to the exact origin of the phenomena observed, it seems desirable that the experimental results should be recorded at the present time, for the full investigation will probably be somewhat prolonged, and the results already arrived at are of interest.

Examination of the photographic effect of the beam reflected from the NS planes—obtained by giving these a small angle with the vertical—revealed no variation in intensity from end to end of the band. There was no indication either of the bands or the fringes running EW, which were so clearly marked in the photographic effects of the pencils reflected from the EW cleavage planes. Thus of the two reflected beams, reflected presumably from the same molecules, one exhibited periodic variation in intensity from end to end, while the other showed no such variation. The latter, however, showed fringes parallel to its own NS reflecting planes. We cannot at present account for this except by interference of one form or another. If it turns out to be due simply to a periodic change of angle of the reflecting planes the interference is of the kind resulting in regular reflection, but shown by this difference in the structure of the images in a more striking way than usual.

The beam of radiation directly transmitted through the crystal, though different portions of it passed through at different angles, produced a photographic image which even when of the same average density as that of the image due to reflection, exhibited absolutely no variation in intensity from end to end. Thus the variation in intensity was in no way due to a variation in absorption of the incident or the reflected radiation.

The source of radiation was moved in a NS plane, so that the angle of incidence on the EW planes changed by about 5 deg. The whole reflected system moved through approximately the same angle. The bands and fringes shown on the photographic plate moved bodily along, the maxima and minima appearing to originate in precisely the same portions of the crystal as previously.

The crystal was moved horizontally in a NS plane over the rectangular aperture, while the source of radiation was fixed. The bands and fringes moved with the crystal, again indicating that maxima and minima of intensity originated in the same positions as previously.

The crystal was rotated about a vertical axis through 180 deg. The features of the band and fringe systems appeared to have turned through 180 deg., the positions being identified by irregularities in the bands and fringes.

Thus the features appeared fixed with respect to the crystal, that is the features *originated* at definite parts of the crystal, at any rate under the stated changes of experimental condition.

Corresponding features were shown, too, in the images obtained by reflection in other vertical planes, as, for instance, the vertical planes running north-west to south-east, and again the maxima and minima appeared to originate in the same parts of the crystal as the corresponding maxima and minima in the other images. In these cases, however, the distance traversed by a ray from one reflecting plane to the next was the same as in the case of the EW planes. The contrast with the image due to reflection in the NS planes was most marked.

As stated above, several crystals were used, and when the other conditions of the experiments were kept constant little variation in fringe-width was observed. A variation in band-width of greater magnitude however was observed, a certain crystal 5 mm. thick, producing bands 25 per cent. smaller than the case of another crystal 2 mm. thick. In these experiments the fringes were not visible, and the bands not so well marked as in later experiments. The possible causes are too numerous to speculate upon at this stage.

The thick crystal was turned through 90 deg. about a vertical axis, so that the previous NS planes become the principal reflecting planes. The band-width was found as closely as measurable the same as in the first position.

The penetrating power of the primary radiation was varied considerably, but when care was taken to keep the other conditions of the experiment as nearly as possible the same, no variation in band-width was observed.

The quality of the radiation in the reflected pencil was tested, first by placing absorbing sheets of aluminium in its path and observing the density of the photographic image. The reflected radiation was found to be of a penetrating type. It was neither corpuscular (electronic) radiation nor characteristic (fluorescent) X-radiation, but might be of the quality of the only other type of secondary radiation known—scattered X-radiation. By the ionisation method* it was found that the radiation constituting the reflected pencil (together with a scattered radiation, the intensity of which was scarcely sufficient to produce appreciable photographic effect) was comparatively homogeneous. The same was true of the directly transmitted pencil of X-radiation, which it closely resembled in penetrating power. It was probably no more homogeneous than any ordinary beam of Röntgen radiation becomes by transmission through a thick sheet of substance by selective absorption of the longer waves.† The intensity of a reflected pencil was measured by comparing the times required by this and by the directly transmitted pencil to produce equal photographic effects. It was found that the intensity of the reflected pencil was only $\frac{1}{1500}$ of that of the directly transmitted pencil. This is only a very small fraction which may be roughly estimated as of the order $\frac{1}{200}$ of the total energy of the radiation scattered by the rock-salt. [We should expect regular reflection to take place from comparatively fixed electrons, whereas scattering might be produced by both fixed and moving electrons.]

Finally, we have not yet arrived at any definite conclusion regarding the exact origin of the periodic variation in intensity of the reflected beam. It appears to be due to a corresponding large structure in the crystal itself. If there were a periodic break in crystalline structure corresponding features might be expected to be shown by reflection in the NS planes as well as by reflection in the EW planes. We have seen no indication of this. Again a periodic progressive change of angle of the cleavage planes would result in a separation of the reflected pencils such as we have not observed. A periodic change of angle alternating in direction would produce two systems of fringes the superposition of which in some positions would produce the fringe systems observed. It will be necessary to carefully trace the directions of these fringes from the crystal source outwards in order to determine their exact method of propagation. Such twinning, however, cannot as far as we see at present account for the separation into bands.

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* W. H. Bragg first showed that the reflected radiation produced ionisation in air, while Moseley and Darwin found in a certain case the penetrating power of the radiation was very like that of the primary radiation.

† We should like here to take the opportunity of correcting some misapprehensions that arose in the discussion on this Paper. One of us said that the *fluorescent X-radiation from an element* was quite probably as homogeneous as the radiation giving some spectral lines—say as homogeneous as the light from a sodium flame; that the homogeneity of the X-radiation here dealt with was, however, only such homogeneity as would result from a selective absorption of the longer waves from a radiation originally giving a continuous spectrum between certain limits. How homogeneous this was we were not then prepared to say.

The fact that the features disappear when the radiation proceeds in certain directions is evidence of interference.

The investigations are being continued. Neither the variation in intensity of the bands nor the origin of the brushes of radiation has yet been studied by us.

NOTE.—I wish to thank the Solvay International Institute for a grant in aid of these researches, and Dr. Sibly for a number of crystal specimens and for his kindness when I have consulted him on crystallography.—C. G. B.

ABSTRACT.

The authors have made a preliminary investigation of the Röntgen radiation proceeding from a crystal of rock salt (which is of the simple cubical form) when a pencil of Röntgen radiation is incident in a direction nearly grazing one of the three sets of mutually perpendicular cleavage planes.

Reflection of X-rays by the Cleavage Planes.—Using a very narrow pencil of radiation, it was seen that the principal secondary pencil was one obeying the laws of reflection from the cleavage planes.

A pencil diverging in all directions from a point source produced a corresponding reflected pencil of radiation converging to a line focus after reflection from a set of parallel cleavage planes. The quality of the radiation forming the secondary pencils was shown both by the photographic and by the ionisation method to be, not the fluorescent X-radiation, but of the kind previously described as scattered X-radiation. It was approximately of the same penetrating power as the primary radiation, and was approximately homogeneous, having traversed 5 mm. of rock salt in the case investigated.

Interference Fringe Systems.—A diverging pencil of radiation was directed on to a crystal so that various portions were incident on the cleavage planes at different angles. A photographic plate showed the relative intensity of the corresponding reflected radiations. It was seen that the intensity of the reflected pencil varied periodically

with varying angle of incidence, the maximum being separated by intervals corresponding to approximately equal increments in the value of cos θ, where θ was the angle of incidence on the reflecting planes.

Such a series of maxima may be explained by interference of the pencils reflected from equal spaced parallel planes, the maxima being spectra of various orders.

The wave-length, calculated on the assumption that these are planes passing through corresponding portions of molecules in the planes of cleavage, and that a molecule is simply NaCl, is found to be $0{\cdot}6 \times 10^{-9}$ cm. If the molecule be more complex, the calculated wave-length would be greater. This value thus agrees remarkably well with the value (between 1 and 2×10^{-9} cm.) calculated from the velocity of ejection of electrons by this X-radiation, taking this to behave as ultra-violet light of short wave-length.

Using a more extended source the above "spectral lines" became indistinct and disappeared, but there appeared a periodic variation in intensity, the band-width being about four times the distance between the above lines. Such a system of interference bands would be produced by a second plane of reflection within the molecules in a position dividing the distance between the other planes in the ratio 1 : 3 approximately. All the evidence considered indicates that these effects are due to interference. Various crystals were used, and one was turned through a right angle so that another system of planes acted as reflecting planes.

The only experiment made up to the time of writing on the effect of varying the penetrating power showed a 25 per cent. smaller band-width with a radiation of increased penetrating power. Further experiments are, however, being made.

Finally, there can be little doubt that the fringe systems are interference fringe systems. That the smaller system is a series of spectra of different orders and the other an interference band system seems probable; this theory certainly explains the results observed up to the time of writing.

DISCUSSION.

The PRESIDENT remarked that Prof. Barkla, in explaining the second system of fringes, assumed that each molecule contained an excentrically placed electron similarly situated, and that all these electrons were stationary, which was contrary to the usual ideas on the subject. Prof. Barkla also rather sharply distinguished between regular reflection and scattering. But regular reflection could be considered to be due to a system of scattered wavelets.

Prof. C. H. LEES thought it was surprising to see that the results were capable of such a simple explanation, and hoped that such experiments would lead to more knowledge of the structure of the molecules.

Prof. J. W. NICHOLSON pointed out that if the molecule had been assumed to be more complex than NaCl the wave-length obtained would have been greater.

Dr. G. W. C. KAYE remarked that these experiments constituted a fresh blow to the corpuscular theory of the X-rays. The present experiments showed a great similarity between X-rays and ordinary light. Had Prof. Barkla used other crystalline substances? He hoped that the results obtained with rock salt could be generalised from.

Mr. C. E. S. PHILLIPS inquired about the time of exposure necessary

Had Prof. Barkla made experiments with mica or star-mica? Some experiments with these substances made at the Cancer R search Hospital showed irregularities they could not easily explain.

Mr. D. OWEN remarked that the experimental results were interpreted by the author as implying the existence of homogeneous etheric waves. It seemed, however, unthinkable that radiation of definite wave-length could be emitted from the anti-cathode of the X-ray bulb, considering that the exciting cause of the production of these rays was the bombardment of the metal by cathode-ray particles. The values deduced for the wave-length were of the same order as the dimensions of the molecule. This result suggested that the definiteness of structure of the rays was imparted by the arrangement of the molecules in the crystal employed, that this definiteness existed only *after* the crystal had produced its effect, and that what was being measured was not the wave-lengths of the incident radiation but the distance apart of the molecules. Might not the phenomena be then of the same kind as those occurring in the "palisade effect" in acoustics? It would be of interest to know whether the wave-lengths found depended on the nature of the X-ray bulb, whether "hard" or "soft."

Prof. BARKLA, in reply, stated that by chemists it was considered highly probable the molecule was more complex than NaCl. On the theory given, the wave-length would be proportional to the cube root of the number of atmos in the molecule; thus, if a molecule were Na_8Cl_8, this would make the calculated wave-length twice as great. When the full anti-cathode was used the exposure given was four or five hours. With a restricted pencil the exposures given sometimes amounted to 18 hours. The crystal plates employed were 2 mm. to 5 mm. thick. After passing through this plate the rays were very homogeneous. One of the things that the corpuscular theory of X-rays had been put forward to explain was the fact that X-rays of a given quality, when absorbed by any material, caused the emission of electrons, the maximum velocity of which depended solely on the quality of the X-rays, and not on their intensity. But the same was true of light, the wave nature of which there was little disposition to question.

NOTE.—The results of further experiments are given in the Paper.

XXII. *Some Oscillograms of Condenser Discharges, and a Simple Theory of Coupled Oscillatory Circuits.* *By* J. A. FLEMING, *M.A., D.Sc., F.R.S.*

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THE theory of the single and of two coupled oscillatory circuits containing capacity and inductance in series is well understood, and has been made the subject of innumerable Papers from the date of Lord Kelvin's first classical memoir and onwards. Of late years it has been treated with especial elaboration with reference to phenomena in wireless telegraphic apparatus. For teaching purposes we require simple methods of arriving at the chief formula and objective means of illustrating the results, so as to give the student confidence in the mathematical deductions. I have found the following method of discussing the phenomena well suited for engineering students, as it makes use of ideas brought before them in other studies and involves no mathematics beyond the simple algebra of complex quantities.

We may discuss as follows first the problem of a condenser discharging through an inductive circuit. To make the problem general, we shall assume the condenser to be a leaky one.

Suppose free damped oscillations have been established in a circuit consisting of a wire having resistance R and inductance L, completing the circuit of a condenser of capacity C having leakance S. Let n be the frequency of these oscillations, and let $2\pi n=p$. If, then, we assume, as we may do, that the currents and voltages vary harmonically, and are proportional to the real part or horizontal step of ε , where $j=\sqrt{-1}$ and $\mathrm{P}=p+ja$, in which a is the damping factor, then the voltage drop down the resistance at the instant when the current is I is $(\mathrm{R}+j\mathrm{PL})\mathrm{I}$. Also the voltage drop down the condenser is $\frac{\mathrm{I}}{\mathrm{S}+j\mathrm{PC}}$, and, since the oscillations are free, there is no impressed E.M.F. Hence the total voltage drop down the whole circuit is zero, or

$$(\mathrm{R}+j\mathrm{PL})+\frac{1}{\mathrm{S}+j\mathrm{PC}}=0. \qquad (1)$$

Therefore $$(\mathrm{R}+j\mathrm{PL})(\mathrm{S}+j\mathrm{PC})=-1=j^2. \qquad (2)$$

Divide all through by CL, and put $2a$ for R/L and $2b$ for S/C. Then

$$(j\mathrm{P}+2a)(j\mathrm{P}+2b)=\frac{j^2}{\mathrm{CL}}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Add to both sides $(a-b)^2$ and rearrange, and we have

$$\left\{j\mathrm{P}+(a+b)\right\}^2=j^2\left\{\frac{1}{\mathrm{CL}}-(a-b)^2\right\}. \quad . \quad . \quad . \quad (4)$$

Taking the square root and multiplying by $-j$ all through we have

$$\mathrm{P}=p+j\alpha=j(a+b)\pm\sqrt{\frac{1}{\mathrm{CL}}-(a-b)^2}, \quad . \quad . \quad . \quad (5)$$

and then equating real parts we have

$$p=2\pi n=\pm\sqrt{\frac{1}{\mathrm{CL}}-\left(\frac{\mathrm{R}}{2\mathrm{L}}-\frac{\mathrm{S}}{2\mathrm{C}}\right)^2}, \quad . \quad . \quad . \quad (6)$$

which gives us the frequency of the oscillations in the circuit. The damping factor $\alpha=(a+b)$ and the ratio of two successive oscillations is $\varepsilon^{-(a+b)\mathrm{T}/2}$; hence the decrement δ of the oscillations per half period is

$$(a+b)\mathrm{T}/2, \text{ or } \delta=\frac{1}{2n}\left(\frac{\mathrm{R}}{2\mathrm{L}}+\frac{\mathrm{S}}{2\mathrm{C}}\right) \quad . \quad . \quad . \quad . \quad (7)$$

If the condenser is assumed to be non-leaky, we have the usual formulæ for the frequency and decrement of a non-radiative or closed circuit thus arrived at in the simplest manner. The equation (6) shows that the frequency depends on the damping and that the rate at which the oscillations decay increases as the resistance R of the circuit or the leakance S of the condenser increases. These formulæ can be confirmed both qualitatively and quantitatively by experiments made with a Duddell oscillograph. For this purpose we attach to the shaft of an ordinary single-phase alternator a commutator consisting of brass segments let into an insulating disk of stabilit, one sector for each pole of the alternator. Three metal gauze brushes press against the sectors and the sectors are staggered, so that whilst the middle brush always rubs on a sector the outside brushes are alternately in metallic contact with the middle brush as the sectors run round. The motor of the oscillograph is driven synchronously direct off the alternator. A condenser of suitable capacity has one terminal connected to the middle brush, and the two outside brushes are connected respectively to a battery of cells and to the condenser discharge circuit. The remaining terminals of the condenser, battery and discharge circuit are connected together. The condenser is then charged and discharged synchronously with the movement of the tilting mirror of the oscillograph. The oscillograph wires can be connected through resistances with the terminals of the condenser or with two points on the discharge circuit.

We have then on the oscillograph screen a visible representation of the oscillation of current or condenser P.D., which can be photographed. The oscillograms in the appended Plate were taken with a condenser of the capacity of about 5 mfd. and with variable resistance in the discharge circuit. They show the slight variation of frequency with resistance, and the rapidly increasing decrement at the resistance R increases. In this case the condenser was non-leaky. The critical case, when $\frac{1}{CL}=\frac{R^2}{4L^2}$, corresponding to cessation of oscillations, is confirmed by the oscillograms Nos. 11, 12, and 13. The value of R, which just annuls oscillation, agrees closely with the value predicted by theory.

Supposing the resistance of the discharge circuit is kept constant and small, so that $R^2/4L^2$ is small, and if the leakance of the condenser is gradually increased by a non-inductive resistance placed as a shunt across the terminals, then there should also be a decrement which increases with the value of S, and finally extinguishes the oscillations when

$$\frac{1}{CL}=\left(\frac{R}{2L}-\frac{S}{2C}\right)^2.$$

This is well shown by the oscillograms Nos. 14 to 21, which illustrate the effect of gradually increasing the leakance of the condenser, and show that the leakance, which just annuls all oscillations, agrees with that predicted by theory.

Turning, then, to the case of coupled oscillatory circuits, we can establish in the same simple manner the essential theory of the interaction.

Let there be two oscillatory circuits having condensers of capacity C_1 and C_2, and leakances S_1 and S_2, in series with circuits of resistance R_1 and R_2 and inductance L_1 and L_2 respectively. Let free oscillations be excited in one circuit and react upon the other, the mutual inductance being M.

Then if these oscillations are damped they can be represented by the real parts of $\varepsilon^{j(p+ja)t}=\varepsilon^{jPt}$, where $p=2\pi$ times the frequency and a is the damping coefficient$=\mathrm{R}/2\mathrm{L}+\mathrm{S}/2\mathrm{C}$. Hence, when the circuits are in oscillation and left to themselves, the currents I_1 and I_2 are determined by the two equations

$$\left.\begin{aligned}\left\{(\mathrm{R}_1+j\mathrm{P}_1\mathrm{L}_1)+\frac{1}{\mathrm{S}_1+j\mathrm{P}_1\mathrm{C}_1}\right\}\mathrm{I}_1+j\mathrm{MP}_2\mathrm{I}_2=0,\\ j\mathrm{MP}_1\mathrm{I}_1+\left\{(\mathrm{R}_2+j\mathrm{P}_2\mathrm{L}_2)+\frac{1}{\mathrm{S}_2+j\mathrm{P}_2\mathrm{C}_2}\right\}\mathrm{I}_2=0.\end{aligned}\right\} \quad . \quad . \quad (8)$$

Writing Tim_1 for the function $\mathrm{R}_1+j\mathrm{P}_1\mathrm{L}_1+(\mathrm{S}_1+j\mathrm{P}_1\mathrm{C}_1)^{-1}$ and Tim_2 for the same function for the secondary circuit, these being respectively the total impedances for damped oscillations, we have

$$\begin{aligned}Tim_1\mathrm{I}_1+j\mathrm{MP}_2\mathrm{I}_2=0,\\ j\mathrm{MP}_1\mathrm{I}_1+Tim_2\mathrm{I}_2=0.\end{aligned} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eliminating I_1 and I_2 the determinant

$$\begin{vmatrix} Tim_1, & j\mathrm{MP}_2 \\ j\mathrm{MP}_1, & Tim_2 \end{vmatrix}=0, \quad . \quad . \quad . \quad . \quad (10)$$

or $Tim_1\ Tim_2+\mathrm{M}^2\mathrm{P}_1\mathrm{P}_2=0$ gives us an equation determining the frequency of the oscillations set up in the circuits.

Suppose we limit ourselves to the case of non-leaky condensers and consider the resistance of the secondary circuit to be small. Then $\mathrm{S}_1=\mathrm{S}_2=0$ and also if $\mathrm{R}_2=0$, we have $\mathrm{P}_2=p=2\pi n$.

Let us denote the mutual inductance by $\mathrm{M}=k\sqrt{\mathrm{L}_1\mathrm{L}_2}$, where k is the coefficient of coupling; then the determinant (10) reduces to

$$\begin{aligned}(1-\mathrm{P}_1{}^2\mathrm{C}_1\mathrm{L}_1+j\mathrm{P}_1\mathrm{C}_1\mathrm{R}_1)(1-\mathrm{P}_2{}^2\mathrm{C}_2\mathrm{L}_2+j\mathrm{P}_2\mathrm{C}_2\mathrm{R}_2)\\ =\mathrm{M}^2\mathrm{P}_1{}^2\mathrm{P}_2{}^2\mathrm{C}_1\mathrm{C}_2, \quad (11)\end{aligned}$$

or, since $\mathrm{P}_2=p$ and $\mathrm{P}_1=p+ja_1$ and $\mathrm{M}^2=k^2\mathrm{L}_1\mathrm{L}_2$, we have

$$\begin{aligned}(1-p^2\mathrm{C}_2\mathrm{L}_2)\Big[1-(p^2-a_1{}^2)\mathrm{C}_1\mathrm{L}_1-j2a_1p\mathrm{C}_1\mathrm{L}_1+jp\mathrm{C}_1\mathrm{R}_1-a_1\mathrm{C}_1\mathrm{R}_1\Big]\\ =k^2(p^2-a_1{}^2)p^2\mathrm{C}_1\mathrm{L}_1\mathrm{C}_2\mathrm{L}_2+jk^22a_1p^3\mathrm{C}_1\mathrm{L}_1\mathrm{C}_2\mathrm{L}_2.\end{aligned}$$

Equating real parts of the above equation and remembering that $a_1=\mathrm{R}_1/2\mathrm{L}_1$, we find at once on substitution that

$$\begin{aligned}1-p^2(\mathrm{C}_1\mathrm{L}+\mathrm{C}_2\mathrm{L}_2)+p^4(1-k^2)\mathrm{C}_1\mathrm{L}_1\mathrm{C}_2\mathrm{L}_2\\ =a_1{}^2(\mathrm{C}_1\mathrm{L}_1-k^2p^2\mathrm{C}_1\mathrm{L}_1\mathrm{C}_2\mathrm{L}_2-p^2\mathrm{C}_1\mathrm{L}_1\mathrm{C}_2\mathrm{L}_2). \quad . \quad (12)\end{aligned}$$

If we assume the circuits are tuned, or that $C_1L_1=C_2L_2=CL$, then equation (12) reduces to

$$\left(p^2\mathrm{CL}(1-k)-1\right)\left(p^2\mathrm{CL}(1+k)-1\right) -a_1{}^2\mathrm{CL}(1-(1+k^2)p^2\mathrm{CL})=0. \quad (13)$$

Suppose, then, that a_1 is very small, or that the damping in the primary is negligible, the equation (13) is seen to be satisfied by

$$p=\frac{1}{\sqrt{\mathrm{CL}(1-k)}} \text{ and by } p=\frac{1}{\sqrt{\mathrm{CL}(1+k)}},$$

and there are, therefore, oscillations of two frequencies given by the above well-known formulæ. If, however, a_1 is very large, or the primary current rapidly quenched by increasing the resistance R_1 to infinity, then the coupling becomes zero when the primary circuit is open, and we have only one oscillation, which is the free oscillation of the secondary circuit. For intermediate values of a there will be three frequencies present. These well-known results are illustrated in the oscillograms Nos. 23 to 40 in the appended Plates which show the effect of gradually increasing the coupling of two inductively coupled circuits, and thus creating in the oscillation trains *beats* indicating the presence of two frequencies. The primary circuit in this case remains closed. If the primary circuit is opened very soon, so as to quench the primary current, then the oscillograms Nos. 32 to 36 show the existence of only a single-period oscillation, which is the free oscillation of the secondary circuit, provided the coupling is not very close. The oscillograms show that even if the two circuits are not tuned when the primary current is suddenly arrested, we have in the secondary an oscillation of frequency given by $p=2\pi n=1/\sqrt{C_2L_2}$, as it should be.

In the case when the coupling is not very loose and the primary current not instantly damped, we may have oscillations of three periods co-existing in the secondary circuit one, its own free oscillation corresponding to the last factor in equation (13), and the other two the periods due to the reaction of the two circuits prior to the instant when the primary resistance becomes infinite. This is well shown in a number of resonance curves described by the Author and Mr. Dyke in a Paper on "Some Resonance Curves taken with Impact and Spark Ball Discharges," "Proc." Phys. Soc., Lond., Vol. XXIII., 1911, p. 136, in which resonance curves with three humps are shown. The function *Tim* above defined is, of course, only an extended form of Heaviside's resistance operator, the application of which in all problems involving oscillation transformers was recently explained by Dr. Eccles in a Paper on the "Application of Heaviside's Resistance Operators to the Theory of the Air-core Transformer and Coupled Circuits in General." (*See* W. H. Eccles, D.Sc., "Proc." Phys. Soc., Lond., Vol. XXIV., 1912, p. 261.)

The function *Tim* may be called the complete resistance operator of a circuit having inductance and capacity, and the whole theory of the transformer is then embraced in the equations (8), (9) and (10). The application to predict the graph of the primary and secondary current is given in Dr. Eccles' Paper above. The oscillograms given in connection with the present Paper have been carefully taken by Mr. Dyke, and as the capacities, inductances and resistances are given in every case, they will serve to check the theory and to enable any student to predetermine the frequencies which will arise without the necessity for anything but the simplest algebraic equations. It is possible from the oscillograms, by measuring the ratio of successive oscillations, to make a fairly good approximation to the calculated decrement from the known resistance, capacity and inductance of the circuit.

The above Paper was illustrated by about 40 oscillograms of condenser discharges of single and coupled circuits prepared as lantern slides.*

They were as follows (*see* Plate) :—

Series I. (*Slides 1 to 5*).—The first set of five slides show the oscillatory discharge of condensers having capacities respectively of 4·0, 2·0, 1·0, 0·5, 0·25 mfd. when discharged with oscillations through an inductive resistance having an inductance of 0·031 henry and a resistance less than 19 ohms. The frequencies corresponding to these various capacities are 450, 640, 900, 1,280 and 1,800, and measurements made on these oscillograms show that the frequency varies inversely as the square root of the capacity, as it should do by theory.

Series II. (*Slides 6 to 10*).—The second set of five slides show the effect of increasing the resistance of the charge circuit, the capacity remaining constant at 5·0 mfd. and the inductance at 0·031 henry. The resistance is successively increased to the values 4·4, 22·3, 51·6, 72·0 and 112·0 ohms, and the effect of this is shown by the progressively increasing damping of the oscillations, the frequency remaining constant.

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* The Cambridge Scientific Instrument Co., of Cambridge, England, have undertaken to reproduce and supply these slides to colleges and teachers who may be desirous of possessing them, but have not the means of preparing the oscillographs themselves.

It should be noted that in the first two slides of this series the particular arrangement used for obtaining the oscillogram does not permit the whole of the oscillations in the train to be photographed; only three or four complete oscillations are shown, but this does not exhaust the energy stored up. It is, however, sufficient to show the damping or ratio of one oscillation to the next.

The decrement δ in each case can be calculated from the formula $\delta = R/4nL$, and can be measured from the ratio of successive amplitudes, since $\delta = \log_e I_1/I_2$.

Corresponding to the various resistance (R) and frequencies (N) the calculated decrements and decrements measured from the oscillograms are as follows :—

R.	N.	δ (calc.).	δ (obs.).
4·4	405	0·07	0·11
22·0	400	0·45	0·38
51·6	363	1·05	1·06
72·0	356	1·63	1·6
112·0	284	3·15	—

Series II.a (*Slides 11 to 13*).—Three slides, forming a continuation of Series II. on an enlarged scale. The well-known formula for the frequency

$$N = \frac{1}{2\pi}\sqrt{\frac{1}{CL} - \frac{R^2}{4L^2}}$$

shows that in the case of a condenser discharged the discharge ceases to be oscillatory when $\frac{1}{CL} = \frac{R^2}{4L^2}$.

The three slides in this series show the observed effects when $\frac{1}{CL}$ is a little greater than, equal to, in a little less than $\frac{R^2}{4L^2}$.

In a condenser circuit the damping depends not only on the resistance of the circuit in series with the condenser, but on any leak in the condenser.

Series II.b (*Slides 14 to 18*).—Consists of five slides showing the effect of gradually diminishing a resistance (r) in parallel with the condenser. In this case the capacity was 5·0 mfd., the inductance of the circuit 0·031 henry and the series resistance about 4 ohms. The shunt resistance was made successively infinity, 278, 120, 86 and 55 ohms, and the oscillograms show the gradually increasing damping. Corresponding to this, the frequency (N) and the observed and calculated decrements are as given below :—

r.	N.	δ (calc.).	δ (obs.).
00	405	0·08	0·07
278	400	0·53	0·57
120	382	1·17	1·24
86	356	1·71	1·63
55	284	3·26	—

Series II.c (*Slides 19 to 22*).—This consists of four slides which are a continuation of Series II.b on an enlarged scale, intended to show that for the shunt conductivity S the discharge just ceases to be oscillatory when $\frac{1}{LC}=\frac{S^2}{4C^2}$. Three of the oscillograms show the state of affairs when $\frac{1}{LC}$ is a little greater than, equal to, or a little less than $\frac{S^2}{4C^2}$, and the fourth when $\frac{1}{\sqrt{LC}}=\frac{R}{2L}=\frac{S}{2C}$, when the discharge again becomes oscillatory, though the damping is large.

Series III. (*Slides 23 to 26*).—Four oscillograms illustrate the reaction between a primary and secondary circuit, both having capacity and inductance, but coupled with various degrees of coupling (mutual inductance).

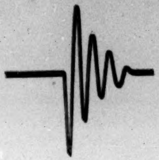
The primary and secondary circuits are tuned so that the product CL is the same for both, and the frequency is 1,610. It is well known that when the coupling is close (as shown on slide 26), then the reaction of the circuits produces in each oscillations of two different frequencies, and these make themselves evident by producing the phenomena of *beats* in the train of oscillations.

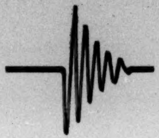
Series IV. (*Slides 27 to 28*).—These two oscillograms are taken with the same coupled circuits, only in the first case the primary current is not quenched, and in the second it is rapidly damped out to show the resulting annihilation of the *beats*.

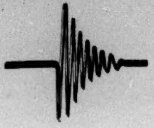
Series V. (*Slides 29 to 34*).—These six oscillograms of the oscillations in two coupled circuits illustrate the effect of varying the resistance, and therefore damping in the two circuits.

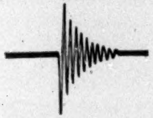
The circuits were tuned and loosely coupled. The slides 29, 30, 31 show the effect of gradually increasing the resistance

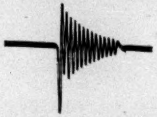
PLATE.

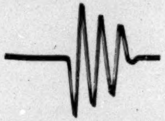

(1) Series I. L=0·031 henry. $R<19$ ohms C=4·0 mfds. $n=450$.

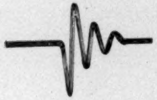

(2) Series I. L=0·031 henry. $R<19$ ohms C=2·0 mfds. $n=640$.

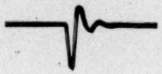

(3) Series I. L=0·031 henry. $R<19$ ohms C=1·0 mfds. $n=900$.

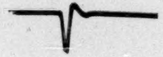

(4) Series I. L=0·031 henry. $R<19$ ohms C=0·5 mfd. $n=1280$.

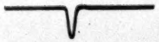

(5) Series I. L=0·031 henry. $R<19$ ohms C=0·25 mfd. $n=1800$.

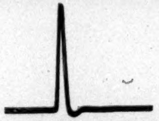

(6) Series II. L=0·031 henry. C=5·0 mfds. R=4·4 ohms.


(7) Series II. L=0·031 henry. C=5·0 mfds. R=22·3 ohms.

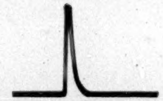

(8) Series II. L=0·031 henry. C=5·0 mfds. R=51·6 ohms.


(9) Series II. L=0·031 henry. C=5·0 mfds. R=72 ohms.

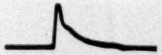

(10) Series II. L=0·031 henry. C=5·0 mfds. R=112 ohms.


(11) Series IIa. L=0·031 hny. C=5·0 mfds. R=112 ohms.

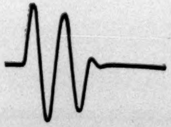
$$\frac{1}{LC}>\frac{R^2}{4L^2}.$$

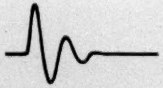

(12) Series IIa. L=0·031 hny. C=5·0 mfds. R=155 ohms.

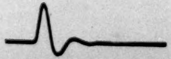
$$\frac{1}{LC}=\frac{R^2}{4L^2}.$$

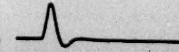

(13) Series IIa. L=0·031 hny. C=5·0 mfds. R=400 ohms.

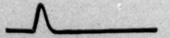
$$\frac{1}{LC}<\frac{R^2}{4L^2}.$$


(14) Series IIb. L=0·031 hny. C=5·0 mfds. R=4·0 ohms. Condenser shunt=∞ ohms.

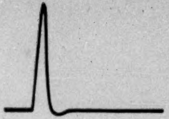

(15) Series IIb. L=0·031 hny. C=5·0 mfds. R=4·0 ohms. Condenser shunt 278 ohms.


(16) Series IIb. L=0·031 hny. C=5·0 mfds. R=4·0 ohms. Condenser shunt=120 ohms.


(17) Series IIb. L=0·031 hny. C=5·0 mfds. R=4·0 ohms. Condenser shunt=86 ohms.

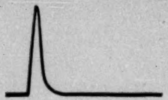

(18) Series IIb.
L=0·031 henry.
C=5·0 mfds.
R=4·0 ohms.
Condenser shunt=55 ohms.

To face p. 224.]


(19) Series IIc. L=0·031 hny. C=5·0 mfds. R=7·0 ohms.

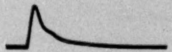
S=Condenser Shunt =55 ohms.

$$\frac{1}{LC}>\frac{S^2}{4C^2}.$$


(20) Series IIc. L=0·031 hny. C=5·0 mfds. R=7·0 ohms.

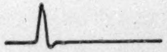
$\frac{1}{S}$=Condenser Shunt =40 ohms.

$$\frac{1}{LC}=\frac{S^2}{4C^2}.$$


(21) Series IIc. L=0·031 hny. C=5·0 mfds. R=7·0 ohms.

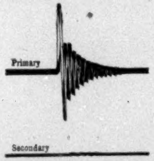
$\frac{1}{S}$=Condenser Shunt =15·5 ohms.

$$\frac{1}{LC}<\frac{S^2}{4C^2}.$$


(22) Series IIc. L=0·031 hny. C=5·0 mfds. R=155 ohms.

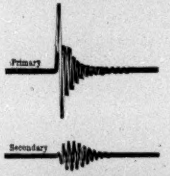
$\frac{1}{S}$=Condenser Shunt =40 ohms.

$$\frac{1}{LC}=\frac{R^2}{4L^2}=\frac{S^2}{4C^2}.$$



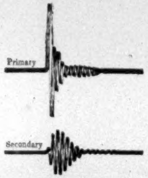
(23) Series III. Zero coupling.

$$\text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=15{\cdot}6\text{ ohms.}\end{cases}\quad \text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=16{\cdot}5\text{ ohms.}\end{cases}$$



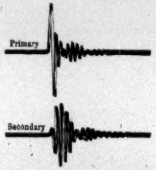
(24) Series III. Loose coupling.

$$\text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=15{\cdot}6\text{ ohms.}\end{cases}\quad \text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=16{\cdot}5\text{ ohms.}\end{cases}$$



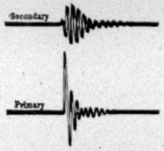
(25) Series III. Moderate Coupling.

$$\text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=15{\cdot}6\text{ ohms.}\end{cases}\quad \text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=16{\cdot}5\text{ ohms.}\end{cases}$$



(26) Series III. Close Coupling.

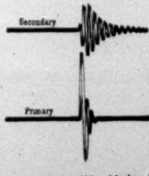
$$\text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=15{\cdot}6\text{ ohms.}\end{cases}\quad \text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C.=0{\cdot}75\text{ mfd.}\\ R=16{\cdot}5\text{ ohms.}\end{cases}$$



(27) Series IV. Moderate Coupling.

$$\text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=16{\cdot}5\text{ ohms.}\end{cases}\quad \text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=15{\cdot}6\text{ ohms.}\end{cases}$$

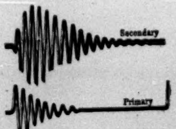
Primary not quenched.



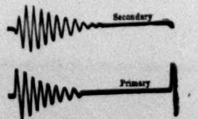
(28) Series IV. Moderate Coupling.

$$\text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=16{\cdot}5\text{ ohms.}\end{cases}\quad \text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}75\text{ mfd.}\\ R=15{\cdot}6\text{ ohms.}\end{cases}$$

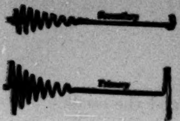
Primary quenched at first. Current node.



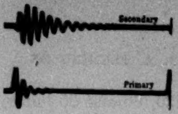
(29) Series V.
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=11·5 ohms.
Pr. R= 5·4 ohms.



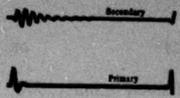
(30) Series V.
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=27 ohms.
Prim. R=5·4 ohms.



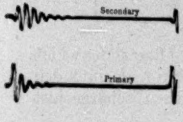
(31) Series V.
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=53 ohms.
Pr. R=5·4 ohms.



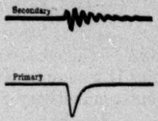
(32) Series V.
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=11·5 ohms.
Prim. R=24·9 ohms.



(33) Series V.
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=11·5 ohms.
Prim. R=88 ohms.

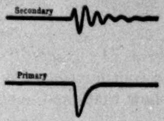


(34) Series V.
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2mfds. L=0·013 hny.
Sec. R=27 ohms.
Pr. R=24·9 ohms.



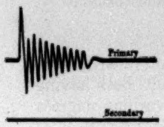
(35) Series VI. Aperiodic Primary, Secondary Tuned.

$$\text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=2{\cdot}0\text{ mfds.}\\ R=11{\cdot}5\text{ ohms.}\end{cases}\quad \text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=2{\cdot}0\text{ mfds.}\\ R=462\text{ ohms.}\end{cases}$$



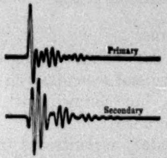
(36) Series VI. Aperiodic Primary, Secondary not Tuned.

$$\text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=4{\cdot}0\text{ mfds.}\\ R=11{\cdot}5\text{ ohms.}\end{cases}\quad \text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=2{\cdot}0\text{ mfds.}\\ R=462\text{ ohms.}\end{cases}$$



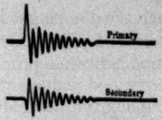
(37) Series VII. Zero coupling Circuits Tuned.

$$\text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}5\text{ mfd.}\\ R=6{\cdot}0\text{ ohms.}\end{cases}\quad \text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}5\text{ mfd.}\\ R=28\text{ ohms.}\end{cases}$$



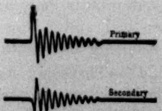
(38) Series VII. Close coupling Circuits Tuned.

$$\text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}5\text{ mfd.}\\ R=6{\cdot}0\text{ ohms.}\end{cases}\quad \text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}5\text{ mfd.}\\ R=28\text{ ohms.}\end{cases}$$



(39) Series VII. Close coupling Circuits Tuned($C_1L_1=C_2L_2$) Secondary Aperiodic.

$$\text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}5\text{ mfd.}\\ R=6{\cdot}0\text{ ohms.}\end{cases}\quad \text{Sec.}\begin{cases}L=0{\cdot}013\text{ ohms.}\\ C=0{\cdot}5\text{ mfd.}\\ R=400\text{ ohms.}\end{cases}$$



(40) Series VII. Close coupling Circuits not Tuned ($C_1L_1\neq C_2L_2$) Secondary Aperiodic.

$$\text{Prim.}\begin{cases}L=0{\cdot}013\text{ henry.}\\ C=0{\cdot}5\text{ mfd.}\\ R=6{\cdot}0\text{ ohms.}\end{cases}\quad \text{Sec.}\begin{cases}L=0{\cdot}013\text{ henry}\\ C=4{\cdot}0\text{ mfds.}\\ R=400\text{ ohms.}\end{cases}$$

in the secondary circuit, whilst that in the primary remains constant and at low value. The primary resistance=5·4 ohms and the secondary resistance=11·5, 27 and 53 ohms. It will be noticed that the secondary never ceases to oscillate before the primary.

The slides 32, 33, 34 show the effect of gradually increasing the resistance in the primary, whilst that in the secondary remains constant.

Series VI. (*Slides 35 to 36*).—In these cases the damping of the primary is increased until it becomes aperiodic, and it is then seen that the oscillations in the secondary are solely determined by the capacity and inductance of the secondary circuit, no matter whether the circuits are tuned or not. In slide 35 the primary and secondary circuits are tuned, and in slide 36 they are not tuned, but the frequency in the secondary is the same—viz., its own free natural frequency in the two cases.

Series VII. (*Slides 37 to 40*).—This comprises four oscillograms showing the effect of an aperiodic secondary circuit. Slide 37 shows the primary oscillations when the secondary is open, slide 38 the result of closely coupling a tuned secondary circuit of small damping. Slides 39 and 40 show the result of increasing the secondary resistance until it becomes a forced oscillation of the same frequency as the primary, no matter how greatly the secondary capacity is varied. The scale of the last two slides is different from that of Nos. 37 and 38.

In conclusion, the Author desires to thank his former assistant, Mr. G. B. Dyke, B.Sc., who skilfully photographed all the above-mentioned oscillograms, and prepared therefrom the lantern slides, which have been reduced in scale for the accompanying Plate.

ABSTRACT.

The author gave a very short method, involving only the simplest algebra, for arriving at a formula for the time of free electrical oscillation of a leaky condenser in series with an inductive resistance, the oscillations being damped. The amplitude of the oscillations is proportional to the real part of ϵ^{Pt}, where $P=p+ja$ and $p=2\pi n$, n being the frequency and a the damping coefficient. Hence the volt drop down the resistance is $R+jPL$, and that down the condenser is $(S+jPC)^{-1}$. The sum of these two is zero, which leads at once to the equation

$$P=p+ja=j\left(\frac{R}{2L}+\frac{S}{2C}\right)\pm\sqrt{\frac{1}{CL}-\left(\frac{R}{2L}-\frac{S}{2C}\right)^2},$$

giving the frequency and damping. The formulæ can be confirmed by oscillograms taken at low frequency with a Duddell oscillograph,

and a number of these were shown demonstrating the accordance of fact with deductions from the formula. In the same manner the case of the coupled circuits was considered and the E.M.F. equations written in the form $Tim_1 I_1 + j M P_2 I_2 = 0$, $j M P_1 I_1 + Tim_2 I_2 = 0$. When Tim stands for $(R + jPL) + (S + jPC)^{-1}$. Eliminating the currents I_1 and I_2, we have an equation which can be solved for p. Taking the reduced case of non-leaky condensers, tuned circuits, and zero resistance in the secondary, Dr. Fleming deduced the equation—

$$\{p^2 CL(1-k)-1\}\{p^2 CL(1+k)-1\} - a_1^2 CL(1-(1+k^2)p^2 CL) = 0,$$

which shows that there are in general, oscillations of three frequencies in the circuits. This was confirmed by photographs of oscillograms and diagrams of resonance curves.

DISCUSSION.

Prof. S. P. THOMPSON admired Dr. Fleming's ingenious method, which was, he considered, much more intelligible to students than writing down the differential equation and then integrating it. He also admired the beautiful oscillograms, and hoped it was possible to procure copies of them, as he considered every teacher ought to have a set.

Dr. W. H. ECCLES also expressed the opinion that Dr. Fleming's method of solution would be valuable to teachers. He also pointed out that the leakage S of the condenser could be so adjusted as to neutralise the effect of the external resistance R on the period, and suggested that this fact might be made use of to measure high frequency resistances.

Prof. G. W. O. HOWE stated that he had used a similar method in teaching. It was very important to clearly bring before students what was happening in the various circuits. The oscillograms could be used to determine the high-frequency resistance, as the log. dec. of the train of oscillations could be measured very accurately.

The AUTHOR, in reply, stated that he possessed the original copies of the oscillograms, and would be pleased to supply copies to teachers desiring them.

XXIII. *An Exhibition of Braun Cathode-Ray Tubes and an Electrostatic Machine for Working them, used as a High-Frequency Oscillograph.* By Dr. J. A. Fleming, *F.R.S.*

Received April 15, 1913. Read March 14, 1913.

For the delineation of extra high-frequency electrical oscillations no mechanical oscillograph is available, and the only possible means is by the aid of a Braun cathode ray tube and phosphorescent screen.

The tubes used by the writer have been made by Messrs. Müller-Uri, of Brunswick, Germany, and have an extra large bulb holding a Willemite screen about 10 cms. in diameter. The tubes are about a metre in length. They have a special cup-shaped cathode and also a vacuum reducing device for letting down the vacuum if the tube becomes too hard. Owing to the sensitiveness of the cathode ray to magnetic fields and to the consequent displacement of the spot of light by the earth's field, it is well to surround the tube between the two perforated screens with a circular coil of wire through which a direct-current is passed. This creates a longitudinal magnetic field, and by giving this coil slight displacements the position of the cathode spot can be adjusted. The tubes contain also electrostatic field plates for producing electric displacement. To delineate a condenser discharge we have to make the cathode ray move with uniform, or nearly uniform, velocity across the screen horizontally and to move vertically with a displacement proportional to the discharge current of the condenser. We must also make the spot repeat its path exactly and rapidly. This is secured as follows: A small motor-driven alternator has a commutator put upon the shaft, having insulated sectors, and three brushes pressing on it as described in the previous Paper for charging at a battery and discharging a condenser through an inductive resistance. The electrostatic deflection plates of the Braun cathode tube are connected directly to the terminals of the alternator, or else to those of an interposed air-core transformer. The cathode ray is thus caused to sweep across the screen, and the amplitude of this motion is made much larger than the width of the screen, so that during the actual time of passage of the spot across the screen its motion is sensibly uniform.

The inductive discharge circuit of the condenser is made to be a coil which is so placed near the cathode tube that it deflects the ray at right angles to the motion due to the electrostatic deflection. The commutator secures that this magnetic deflection will always take place just at the same instant when the cathode spot is passing across the screen. If these arrangements are properly made the cathode spot traverses the same path again and again, and the oscillogram is therefore visible as a permanent bright line.

To operate the cathode tube we need therefore some perfectly steady source of high potential. An ideal source is a storage battery of 10,000 or 20,000 small cells. This, however, is difficult to obtain and expensive. The next best appliance is a suitable electrostatic influence machine. The one used by the author, here exhibited, was made by Müller-Uri, of Brunswick, Germany, and is a two-plate influence machine of the Wimshurst type. The plates are of ebonite, and one is fixed and the other revolving. The ebonite plates have the metal sectors embedded in them, or rather sandwiched, between two plates, and communication with them established through small metal buttons or pins which appear on the surface. The revolving plate is driven by a small electric motor. This machine gives a very steady potential. It will give an 8 in. spark in air when in good order and will supply a current sufficient to decompose acidulated water. The makers claim that it will produce a current of 300 to 350 microamperes. It easily deflects a suitable galvanometer and can therefore be used to demonstrate the current-giving power of an electrostatic influence machine. The ebonite plates are kept in good order if rubbed occasionally with a mixture of powdered French chalk and absolute alcohol. They should, of course, be kept covered and not exposed to light.

This machine gives a very steady cathode spot on the phosphorescent screen, and, in conjunction with the arrangements above described, affords a means of visibly representing oscillations of much higher frequency than is possible with any mechanical oscillograph. At the present time there is a great necessity for some form of fairly portable oscillograph, which shall enable us to see the trains of oscillations in a radiotelegraphic transmitter and make an ocular inspection of the decrement possible. The only chance of doing this is by some form of cathode ray tube operated by an electrostatic machine or multicellular battery as above described, but at present the whole collection of apparatus required for this purpose is somewhat elaborate and not very portable.

As far as the experiments have gone, however, they seem to promise a means of doing it with certain appliances; and, further, work on the same subjects is being continued in our radiotelegraphic laboratory in University College.

ABSTRACT.

Some Braun cathode ray tubes used as high-frequency oscillographs, and also an electrostatic influence machine, giving a steady current of 300 to 350 microamperes for working them, were exhibited and described. The Braun tubes have electrostatic deflection plates in them and an embracing field coil for providing a longitudinal field to keep the cathode spot in a central position on the screen. The tubes are worked by connecting the deflection plates to the terminals of an alternator, whereby the cathode spot is made to move to and fro over the screen with a nearly uniform velocity. The shaft of the alternator carries a commutator by means of which a condenser is repeatedly discharged through an inductive coil which acts as a deflecting magnet on the oscillograph tube and displaces the cathode ray at right angles to the direction of the electrostatic displacement. In this way the spot has two motions, a uniform horizontal motion and a vertical oscillatory motion, the two being kept in step with each other. Hence a visual representation is given on the phosphorescent screen of the nature of the oscillation. The electrical machine above mentioned, made by Messrs. Müller-Uri, of Braunschweig, was shown in operation, and the current given by it was seen to be sufficient to cause decomposition of acidulated water in a lantern voltameter.

DISCUSSION.

Mr. A. Campbell stated that one German instrument maker had recently described in a Paper a method of improving the surface of ebonite. The ebonite was painted with a composition termed Bakelite and then heated to nearly the softening point of the ebonite, when the composition spread over the surface and produced a surface like amber, which was unaffected by either sunlight or moisture.

Prof. J. T. Morris stated that he had employed Braun tubes 10 years ago to find the maximum value of an alternating current, but found many difficulties in their use. He would like to know what was the actual voltage on the tube, and also whether the oscillograms were in a continuous line or were made up of a row of dots showing the cathode rays to be produced intermittently, as he believed was often the case with a Wimshurst machine. His brother, Mr. D. K. Morris, had suggested, several years ago, using a different number of sectors on the two plates in order to overcome this. He would like to know whether anyone had tried it. He asked if a high-potential battery would not give a more satisfactory result than the Wimshurst.

Dr. R. S. Willows asked if Dr. Fleming had used a hot lime cathode, as then the tubes could be run from 200-volt mains, and, as the cathode rays so produced had a much smaller velocity, much lower voltages could be used in the condenser discharges.

The Author stated that batteries would be ideal, but their price was prohibitive. Ordinarily Wimshurst machines do give a spotted appearance to oscillograms taken with Braun tubes, but the present machine gave a perfectly uniform and continuous curve. He had never tried hot lime cathodes.

XXIV. *Note on a Method of Observing the Flame Spectra of Halogen Salts.* *By* E. N. DA C. ANDRADE, *B.Sc., Ph.D.,* 1851 *Exhibition Scholar of the University of London.*

RECEIVED MARCH 31, 1913. READ MARCH 14, 1913.

§ 1. It has been shown by Smithells* that if chlorine be introduced into a flame containing the luminous vapours of certain metals—*e.g.*, lithium or strontium—it destroys the coloration. To demonstrate this it is best to introduce the salt by means of a Gouy sprayer; the effect can then be shown by passing the air supplied to the burner over chloroform. The effect cannot be shown by introducing a bead of the salt into the chlorinated flame, because the great volatility of the chloride formed causes the metal to vaporise so much faster that the chlorine is no longer present in sufficient excess to ensure the greater part of the vaporised metal being chemically combined. If the salt be sprayed, however, the amount of metallic vapour present is not affected by the introduction of the chlorine, which then completely extinguishes the colour.

§ 2. When introducing a bead of salt into a chlorinated flame on a nickel wire, I noticed that the nickel wire coloured the flame independently of the presence of any salt. This led me to introduce wires of other metals into the chlorinated flame, in the hope, afterwards justified, of obtaining characteristic spectra. The effect is very striking with copper. An ordinary copper wire in the chlorinated flame gives an intense blue in the lower part of the flame and a strong red in the upper part; the spectrum turns out to be the same as that obtained in another way for copper chloride by Smithells,† and investigated in detail by him. The method described allows the different emissions in the different zones of the flame to be shown remarkably clearly.

§ 3. I divide the flame into four parts: the edge E, the outer mantle M_1, the inner mantle M_2, and the cone C (Fig. 1). Copper gives in the edge a red coloration, in the outer mantle blue; the inner mantle is in general uncoloured. If the wire be held in the cone, however, a streak of blue vapour rises from it into the inner mantel. Holding the wire through the flame just touching the cone, all these colorations can be seen at once in their respective zones, the division being very clearly marked.

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* Smithells, Dawson and Wilson, "Phil. Trans.," 193A, 1900, p. 121. There is here a reference to a previous Paper by Smithells which contains a brief remark on the point in question.

† Phil. Mag." (V.), 39, p. 122, 1895.

The other chloride spectra which I have so far observed are those of nickel, cobalt and iron, the flame spectra of which do not seem to have been fully observed before in the cases where they have been observed at all. Introducing a nickel wire into the chlorinated flame I find in the edge a green coloration, in the outer mantle lavender, from the cone pink.*

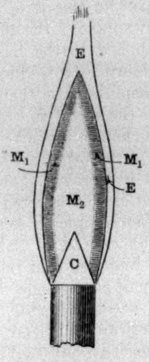

FIG. 1.

Not having cobalt wire, I introduced the glowed salt, which gives no colour in the ordinary flame, on a platinum wire. The flame is then red in the edge and a bluish pink in the outer mantle. Iron gives a yellow coloration, not due to traces of sodium.

§ 4. By heating the wires electrically while they are in the flame it can be shown that the emissions are probably not temperature effects. For instance, the wires may be heated electrically to melting in the edge of the flame without changing the character of the emission ; its intensity increases somewhat with rise of temperature, however.

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* P. J. Hartog (British Assn. "Report," 1901, p. 613) makes mention of a red coloration from the cone, and a *temporary* purple which flashes out and disappears.

§ 5. The chloride spectra obtained in the manner described have certain general characteristics. In all cases there is a continuous background; further, the spectra are banded spectra. Copper chloride, as is well known, shows brilliant bands in the blue and green with a faint continuous background; the green bands are due to the oxide, according to Smithells (*loc. cit.*). The red coloration in the edge gives a continuous spectrum. With nickel the green in the edge is continuous; we have for the spectra in other parts of the flame a continuous background, with bands in the red, blue-green and violet. The red bands are strong in the cone and cause the pink tint. With cobalt there is a red continuous spectrum in the vapour in the edge of the flame, a continuous spectrum with bands in the green and blue superposed on it for other parts of the flame. With iron, apparently in any part of the flame, we have a continuous background with bands in the red, green and yellow, the strongest being the yellow band. It seems that for the different metals the continuous background has its maximum intensity in different places characteristic of the particular metal, as Lenard* has shown to be the case for the continuous backgrounds of the alkali metals. The fact that all these spectra are banded accords with the view that banded spectra may be attributable to molecules, line spectra to atoms.

Attempts have been made to get chloride spectra of tin, lead and mercury by bringing the vapours of these metals into the flame. Characteristic colorations are produced, but at present have not been resolved spectroscopically.

§ 6. I have made some experiments with flames into which the vapours of bromine and iodine respectively have been introduced. A very small amount of bromine vapour is sufficient to extinguish completely the strong colour produced by sprayed solutions of lithium salts. Iodine vapour produces the same effect. Wires of some of the metals already mentioned give characteristic spectra in these flames; these await further examination.

§ 7. A few observations have been made on the deviation in an electric field of the luminous streak of chloride vapour. Lenard† showed in 1902 that the luminous vapour from a bead of salt in a flame is deviated in an electric field, the amount of deviation depending on the metal (and, of course, on the upward velocity of the flame gases, and the strength of the field). G. Ebert in 1911 showed that in the case of strontium this deviation decreased when the vapour was decolorised by chlorine in the flame, tending to become zero as the intensity of coloration decreased. Lenard remarked that copper chloride is deviated in the electric field; he apparently used a bead of the salt, and does not mention the different colorations. From present observation it seems that the red colour is deviated partly to the positive and partly to the negative electrode. The blue goes mainly to the negative electrode. The deviation is, however, especially marked in the blue vapour from the cone, the vapour behaving as if strongly positively charged. Similar results are found for nickel. There is no doubt that the luminous chloride vapour is strongly charged, the carriers (which emit band spectra) being mainly positive, but negative also existing.* Very rough estimates of the velocity of migration of the carriers show that the carriers probably alternate the positive state with the neutral, as assumed by Lenard for the carriers of the line spectra.† As we do not yet know in which state the carriers actually emit the light, we cannot, however, confirm or contradict Stark's assumption that the carriers of the band spectra are neutral.‡

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* "Annalen der Physik" (IV.), 17, p. 208, 1905.

† "Annalen der Physik" (IV.), 9, p. 642.

§ 8. This preliminary note calls attention to the two classes of metallic chlorides :—

1. Those which are non-luminous in the flame; this class apparently consisting of the alkaline and alkaline earth metals.

2. Those which give characteristic banded spectra with continuous grounds.

A new modification of method has been described for obtaining halogen compound spectra of some metals, which enables us to examine with ease the emission of the different flame zones, and has led in some cases to spectra apparently new. The electrical behaviour can also be easily examined by this method. It has been pointed out that, while the vapours of non-luminous chlorides are unchanged, the luminous chloride vapours are strongly charged. I hope to study in more detail the structure of the spectra of the different halogen salts, and the electrical behaviour of the vapours, with a view to getting information on the mechanism of the emission of banded spectra.

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* *Cf.* E. N. da C. Andrade, "Phil. Mag.," July, 1912, p. 16.

† *Loc. cit.*, 1902.

‡ *See* "Atomdynamik," Vol. II., p. 138.

ABSTRACT.

If a flame containing a large amount of chlorine be prepared by passing the air supplied to a colourless gas flame over chloroform, wires of certain metals—copper, nickel, iron, for instance—held in the flame give characteristic colorations in the different zones of the flame. These are due to the chlorides of the metal, which can exist undissociated, in some zones of the flame at least, in the presence of excess chlorine. The chloride spectrum of copper is well known (of Smithells) but the chloride spectra of nickel, cobalt, and iron chloride do not seem to have been fully observed before. The method makes it easy to observe the different emissions which take place in the different zones, and also the electrical migration of the vapour discovered by Lenard in 1902. All the chloride spectra have certain common characteristics. Attention is called to the fact, discovered by Smithells, that for the vapours of some metals—*e.g.*, lithium, strontium—the coloration produced in the flame is destroyed by chlorine. In such cases the vapours are not electrically charged, while in the case of the metallic chlorides which give characteristic spectra in the flame the vapours are strongly charged.

By bringing wires into flames containing bromine and iodine compound spectra were observed in some cases.

DISCUSSION.

Mr. A. Campbell asked if Dr. Andrade could give an explanation of the blue coloured flame obtained when NaCl is thrown on to a fire.

The Author replied that the flame was the spectrum of copper chloride, there being a sufficient impurity of copper in the coal.

XXV. *On the Stretching and Breaking of Sodium and Potassium.*

By BEVAN B. BAKER, *B.Sc., University College, London.*

RECEIVED APRIL 1, 1913. READ MARCH 14, 1913.

THE nature of the structure of solid bodies, and in particular of metals, is a subject which, in spite of its importance, is rather obscure and difficult to determine. The present investigation will, it is hoped, throw a little light on some aspects of this elusive question.

Preliminary Investigation.—Some metals, when in the form of wires, on being stretched break off suddenly at one point, whereas others, plastic metals, thin down and finally collapse to a point at the place where they break.

Some months ago I had occasion to experiment on the behaviour of wires of metallic sodium when in a state of tension. The wires were formed by squeezing the metal through a round hole of about 2 mm. diameter into a bath of pure paraffin oil, to preserve the wire from oxidation. The wire, so prepared, was found not to collapse to a point nor to break off suddenly on stretching, but to collapse from two opposite sides only, into a chisel end.

To test whether this effect was due at all to the method by which the wires were constructed it was decided to form the wires by melting the sodium and allowing it to solidify in a glass tube.

Method of Forming the Wire.—A glass tube, whose internal diameter was that of the wire required, was sealed on to a much wider tube at one end, and at the other was connected by rubber tubing with an ordinary glass tap (*see* Fig. 1). The tube so formed was filled almost entirely with pure paraffin oil, which had been carefully dried, and the tap was closed. Lumps of metallic sodium were scraped free from oxide under oil and were introduced into the wider part of the tube. The tube was then placed inside a cylindrical electric furnace arranged in a vertical position, so that almost all of the narrow tubing and about half of the wider tubing was inside the furnace. The furnace was heated to about 180°C. and was kept at that temperature for a short time. The tap was then opened slightly to allow the oil to run slowly out of the tube; at the same time the now molten sodium ran down into the narrow tube. When sufficient oil had run out the tap was closed and the whole apparatus was allowed to cool. When cold the tube was removed from the furnace and the tap and rubber tubing were taken off. By introducing a rod, which just fitted the inside of the narrow glass tubing, the mass of solid sodium was easily pushed out into a large bath of paraffin oil.

In this way it was found possible, in a simple manner, to form wires of considerable length and having a remarkably smooth surface.

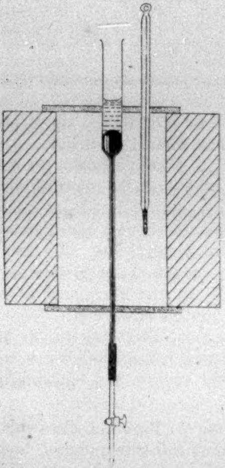

FIG. 1.

Description of the Phenomenon.—When the wire prepared in this way was stretched, it was found not to break off almost directly in one place, as the wire made by squeezing had done, but to thin, for some time, quite uniformly all along its length. This thinning was not, however, a collapse from all sides, but exhibited the same phenomenon of collapsing only from two opposite sides that the other wire had shown. For further extension the wire gave way in certain places and broke in the chisel form.

Moreover, for wires prepared in this way a still further phenomenon became apparent which was scarcely observable for wires made by squeezing. As the wire was extended there began to be apparent on the surface markings in the form of two sets of equidistant rings encircling the wire, crossing one another and lying in planes sloped at an angle of about 45 deg. to the axis of the wire. Members of the two sets touched along the line of greatest thinning, and bisected each other along the line where no thinning took place (*see* plate 1). As the wire was still further stretched up to the breaking point the rings remained unchanged in position or slope, the only difference being that they became more clearly marked. It is noteworthy that usually one set of rings was much more distinct than the other. Microscopic examination made it clear that the rings were actual ridges on the surface.

Some experiments were also made in exactly the same way with metallic potassium, and it was found to contract and form rings precisely like sodium ; there was no apparent difference in their behaviour.

Dr. Andrade had independently noticed that wires of solid mercury when stretched contract to a chisel point. On observing the rings formed on sodium wires he searched for similar rings on mercury, and found that they were formed, only that they were very much finer and closer together than in the case of sodium and potassium.

Explanation of the Effect.—The phenomenon thus appears to show an asymmetric structure for the metals sodium, potassium and mercury. They appear to exhibit the behaviour of plastic metals in one direction and of brittle metals in the direction at right angles.

The effect suggests that the effective portions of the metal may be in the form of cubes set so that two opposite edges are horizontal and in the same vertical plane. When vertical extension of the wire takes place, and there arises a corresponding tendency for lateral contraction, it is clear that the faces inclined at 45 deg. may slip over the similar faces of adjacent crystals, and thus allow of a rearrangement in a narrower space ; on the other hand, the vertical faces can only slip over similar vertical faces, and clearly no contraction in the direction at right angles to such faces can take place.

It is clear that the crystals would pack as closely together as possible, and would form in layers which would lie so as to make with the axis of the wire an angle of 45 deg. The rings observed on the wire would then be due to the edges of the crystals protruding at the surface of the wire

Description of Plate.

(*a*) Wire of sodium of 3 mm. diameter drawn out to breaking. View of specimen in the plane in which no contraction takes place.

(*b*) Same specimen as (*a*). View of specimen in the plane in which maximum contraction takes place.

ABSTRACT.

The author described how wires made of metallic sodium and potassium collapse when stretched, not to a point, as is the case with most plastic substances, but from two opposite sides only, into a chisel end.

The wires upon which experiments were conducted were made in two ways—firstly, by pressing the metal through a small hole into a bath of paraffin oil to hinder oxidation ; and, secondly, by running the metal, molten under oil, into a glass tube and allowing it to solidify. Wires made by both the above methods showed the same behaviour on stretching.

Wires made by the second process also showed, on extension, two sets of equidistant rings on their surface, each inclined at an angle of 45 deg. to the axis, the rings of opposite sets touching along the line of greatest thinning and bisecting one another along the line at which no thinning takes place.

Dr. Andrade has also noticed the same effects of breaking and forming rings with wires made of solid mercury.

The author suggested an explanation of the phenomenon, based on the assumption that the portions of the metal brought into play are in the form of cubes. Such cubes when placed so that a plane through two opposite edges was parallel to the axis of the wire would allow of lateral contraction by faces sliding over one another in one direction only and not in the direction at right angles.

DISCUSSION.

Dr. L. N. G. Filon asked if the section of the stretched wire was accurately elliptical. It was interesting to note that the angle of the rings did not change when the wire was drawn out.

Prof. A. W. Porter remarked that it looked as if the metal possessed a lattice-like structure.

The Author stated that he had not measured the sections to see if they were accurately elliptical.

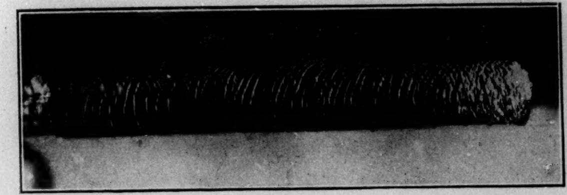

Fig. (*a*).

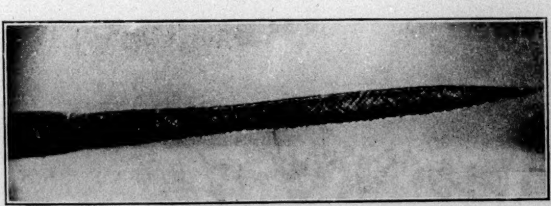

Fig. (*b*).

To face p. 238.]

XXVI. *Note on Optical Imagery.** *By* T. Smith, *B.A.*, *National Physical Laboratory.*

Mr. T. Smith (communicated remarks) stated that he had been interested in looking through Mr. Bower's Paper, but he did not entirely agree that the treatment suggested therein was the best one to adopt with beginners. By starting with the velocity ratio definition of μ it was true that a physical conception of its meaning was obtained, but there was the disadvantage that a pupil could not determine it for himself with the aid of the simple apparatus that would be at his disposal. The better way appeared to be the older one of defining the refractive index as the ratio of sines after the pupil had found experimentally that this ratio was constant; he should be able to find for himself all the laws of refraction. Many assumptions were made in the velocity ratio definition which would not be realised by the student. For instance, he would not easily understand the necessity for an experimental basis for the law relating to the multiplication of refractive indices—$\mu_{A-C}=\mu_{A-B}\times\mu_{B-C}$. The statement of laws in such a way as to veil the assumption of facts which could easily be verified experimentally was to be deprecated. Moreover, when the velocity ratio conception was used the more obvious method to follow with a view to enabling the student to visualise what was taking place was to draw successive wave fronts, since it was easy to show that the effect on a wave front of light of encountering a refracting or reflecting surface of the dimensions with which Geometrical Optics dealt was to alter its curvature by a constant amount when the incidence was normal or nearly normal.

For tracing through a lens system rays to which the conventions of geometrical optics did not apply, it seemed to him best to use Young's construction to determine the directions of the rays, and to calculate in turn for each surface the positions of the primary and secondary focal lines from the algebraic formulæ. These formulæ were most simply derived from the algebraic conditions that the length of the optical path between the object and the so-called image for the chief ray should be equal to that for a slightly displaced ray: each of the formulæ in question could be proved in a couple of lines in this way, and would be more easily followed by a student than the geometrical proofs.

In paragraph 18 (vii.) the author used the term "image" as applied to the point I. In such a position as was shown in Fig. 6 "image" could not have any definite meaning until a special one had been assigned to it: the ordinary definition of geometrical optics did not apply in this case. In the absence of a special definition it appeared desirable to avoid using the word in the presence of aberrations.

He was not inclined to agree with the author in allowing more than was necessary of a pupil's attention to be given to what happened to a ray in its passage between the first and final media of a refracting system. The great advantage of the Gaussian method of treating lens systems was that all systems with a finite focal length might be investigated at one and the same time, and the whole subject could be fully treated in a few pages. The present-day text-books made the subject much more lengthy and difficult than was necessary by considering first the case of a simple (thin) lens, to which, though of little practical importance, the attention of pupils was chiefly directed. Then followed thick lenses and afterwards more complex cases. As a result of dividing the subject up into so many different sections, which were treated independently, the student usually failed to grasp any part but the thin lens, and in examinations avoided simple questions on thick lenses. It appeared a wiser course to master at first the very small amount of algebra necessary to treat the general case at the beginning of the subject, and to scrap the special formulæ dealing with individual cases which were of no practical importance. This would help to get rid of much misconception on geometrical optics. For instance, it would help the student to realise that the focal length of a lens was not the distance to the principal focus from the external surface of the lens, and that, because the thickness of the glass of a simple lens was quite small compared with the focal length, the formulæ for a "thin" lens did not necessarily give a fairly accurate value for the focal length. Consider, for instance, a lens of refractive index 1·5 placed in air of thickness 1 cm. and radii of curvature 100 cm. and 99 cm., the centres of curvature being coincident. The focal length is 29,700 cm., which is quite large compared with the thickness 1 cm. Yet the principal points [are not near the surfaces but at the centre of curvature. The formula for thin lenses would make the focal length 19,800 cm. It should be noted that this was not at all an extreme case, but that the curves were very flat, the thickness being small compared with either radius of curvature, and in fact the flatter the curves the worse the approximation given, assuming that the properties of thin lenses held. It was instructive to slightly alter the curvature of one of the surfaces to make the lens one of long positive focal length and determine the distances of the principal foci from the surfaces.

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* A contribution to the discussion on Mr. W. R. Bower's Paper on pp. 160-177 of the present volume.

For any other purposes than those already mentioned, viz., to give a visual impression of what was taking place at a refracting or reflecting surface, and for graphically tracing rays through a system, the algebraic methods seemed to him much simpler than any geometrical methods were likely to be. It would be easy to put these into a form in which the positions of all the cardinal points with respect to the extreme surfaces and the focal lengths could be calculated for a combination of quite a large number of surfaces in a few minutes. He had ventured to put down below the forms which he had found most convenient. It was assumed that there were n surfaces identified by being numbered from 1 to n: the refractive indices are $\mu_0, \mu_1 \dots \mu_n$: let ${}_p = \frac{\mu_p - \mu_{p-1}}{r_p}$ = the power of the pth surface: $K_{a,b}$ = the power of the system of surfaces bounded by the ath and bth inclusive: t_p is the axial thickness between the surfaces bounding the medium μ_p.

$$K_{a,b} = 0 \text{ if } a > b \text{ and } K_{a,a} = \kappa_a.$$

Then

$$\left.\begin{aligned} K_{1,p} &= K_{1,p-1} + \kappa_p \frac{\partial K_{1,p}}{\partial \kappa_p} \\ \frac{\partial K_{1,p}}{\partial \kappa_p} &= \frac{\partial K_{1,p-1}}{\partial \kappa_{p-1}} - \frac{t_{p-1}}{\mu_{p-1}} K_{1,p-1} \end{aligned}\right\} \dots\dots \quad (1)$$

It would easily be seen that by using the formulæ

$$\left.\begin{aligned} K_{p,n} &= K_{p+1,n} + \kappa_p \frac{K_{p,n}}{\kappa} \\ \frac{K_{p,n}}{\partial \kappa_p} &= \frac{\partial K_{p+1,n}}{\partial \kappa_{p+1}} - \frac{t_p}{\mu_p} K_{p+1,n} \end{aligned}\right\} \dots\dots \quad (2)$$

the values of $K_{1,n}$ given by (1) and (2) would be the same, and that

$$\frac{\zeta_1}{\partial \kappa_1} \cdot \frac{\partial K_{1,n}}{\partial \kappa_n} - K_{1,n} \frac{\partial^2 K_{1,n}}{\partial \kappa_1 \partial \kappa_n} = 1. \dots\dots \quad (3)$$

Let (x_0, y_0) be the co-ordinates of a point on an incident ray inclined at θ_0 to the axis, x_0 being measured along the axis from the vertex of the last surface, and y_0 perpendicular to the axis. Similarly let $(x_n y_n)$ be the co-ordinates of a point on the refracted ray inclined at θ_n to the axis, x_n being measured from the vertex of the last surface and y_n perpendicular to the axis. The law of refraction would at once give

$$\mu_n\theta_n=\mu_0\,\theta_0\frac{\partial \mathrm{K}_{1,n}}{\partial\kappa_1}+(x_0\theta_0-y_0)\mathrm{K}_{1,n}\quad\ldots\ldots\quad(4)$$

and

$$x_n\theta_n-y_n=\mu_0\theta_0\frac{\partial^2\mathrm{K}_{1,n}}{\partial\kappa_1\,\partial\kappa_n}+(x_0\theta_0-y_0)\frac{\partial\mathrm{K}_{1,n}}{\partial\kappa_n}.\quad\ldots\ldots\quad(5)$$

The relation (3) would at once give from (4) and (5)

$$\mu_0\theta_0=\mu_n\theta_n\frac{\partial\mathrm{K}_{1,n}}{\partial\kappa_n}-(x_n\theta_n-y_n)\mathrm{K}_{1,n}\quad\ldots\ldots\quad(6)$$

and

$$x_0\theta_0-y_0=-\mu_n\theta_n\frac{\partial^2\mathrm{K}_{1,n}}{\partial\kappa_1\partial\kappa_n}+(x_n\theta_n-y_n)\frac{\partial\mathrm{K}_{1,n}}{\partial\kappa_1},\quad\ldots\ldots\quad(7)$$

results which could be written down from symmetry with (4) and (5).

Equations (4) and (6) contain the whole theory of co-axial lens systems, however complicated, which have a finite focal length. For instance, considering points on the axis, it will be seen that all rays starting from the point $x_0=-\mu_0\dfrac{\frac{\partial\mathrm{K}_{1,n}}{\partial\kappa_1}}{\mathrm{K}_{1,n}}$ emerge parallel to the axis, so that this point is the first principal focus. Similarly the position of the second principal focus is given by

$$x_n=\frac{\mu_n\frac{\partial\mathrm{K}_{1,n}}{\partial\kappa_n}}{\mathrm{K}_{1,n}}$$

Putting x_0 and x_n equal to these values, the equations (4) and (6) become

$$\mu_0\theta_0=-y_0\mathrm{K}_{1,n},$$

and

$$\mu_0\theta_0=y_n\mathrm{K}_{1,n},$$

showing that $\dfrac{\mu_0}{\mathrm{K}_{1,n}}$ and $\dfrac{\mu_n}{\mathrm{K}_{1,n}}$ measure the scale on which the image of an infinitely distant object would be formed. These quantities are called focal lengths. Again the points on the axis for which

$$x_0=\frac{\mu_n-\mu_0\frac{\partial\mathrm{K}_{1,n}}{\partial\kappa_1}}{\mathrm{K}_{1,n}},$$

and

$$x_n=\frac{\mu_n\frac{\partial\mathrm{K}_{1,n}}{\partial\kappa_1}-\mu_0}{\mathrm{K}_{1,n}}$$

have the property that any ray passing through one of them emerges in the same direction as that in which it is incident, and they are called nodal points.

When

$$x_0=\mu_0\frac{\frac{1}{m}-\frac{\partial\mathrm{K}_{1,n}}{\partial\kappa_1}}{\mathrm{K}_{1,n}}\quad\text{and}\quad x_n=\mu_n\frac{\frac{\partial\mathrm{K}_{1,n}}{\partial\kappa_n}-m}{\mathrm{K}_{1,n}}\quad\ldots\ldots\quad(8)$$

we obtain in both cases for points on the axis

$$\mu_0\theta_0=m\mu_n\theta_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

showing that these points are in the relation of object and image.

Putting in these values of x_0 and x_n when the points considered are off the axis we obtain

$$y_n\mathrm{K}_{1,n}=\mu_0\theta_0+m\mu_n\theta\ =my_0\mathrm{K}_{1,n}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

showing that for the object and image planes defined by (8) the linear magnification is m. Putting $m=1$, we get the equations for the unit planes

$$x_0=\mu_0\frac{1-\dfrac{\partial\mathrm{K}_{1,n}}{\partial\kappa_1}}{\mathrm{K}_{1,n}}$$

and

$$x_n=\mu_n\frac{\dfrac{\partial\mathrm{K}_{1,n}}{\partial\kappa_n}-1}{\mathrm{K}_{1,n}}.$$

It is evident that all the usual relations can be quickly obtained by change of origin or by other simple devices. For instance, from (9) and (10) we have $\mu_0\theta_0y_0=\mu_n\theta_ny_n$, a result of great practical importance which is too often neglected.

This method presents no difficulties when the fundamental algebra of (1) and (2) is mastered. It may be as well to illustrate how practical calculations of focal lengths, &c., are best arranged. The following constructional data are very approximately those given in a patent specification for a photographic lens; we may assume that the unit of length is 1 in. The radius of curvature is positive when the surface is convex to the incident light:—

Surface.	μ	r	t
	1		
1		+3	
	1·53		0·2
2		−30	
	1		0·8
3		−4	
	1·60		0·1
4		+4	
	1		0·9
5		∞	
	1·61		0·2
6		$-3\frac{1}{3}$	

Thus we have

$\kappa_1=+0{\cdot}177$ $\quad t_1/\mu_1=0{\cdot}131$
$\kappa_2=+0{\cdot}018$ $\quad t_2/\mu_2=0{\cdot}8$
$\kappa_3=-0{\cdot}15$ $\quad t_3/\mu_3=0{\cdot}063$
$\kappa_4=-0{\cdot}15$ $\quad t_4/\mu_4=0{\cdot}9$
$\kappa_5=0$ $\quad t_5/\mu_5=0{\cdot}134$
$\kappa_6=+0{\cdot}183$

Then from equations (1)

——	$\frac{\partial K_{1,p}}{\partial \kappa_p}$.	$\kappa_p \frac{\partial K_{1,p}}{\partial \kappa_p}$.	$K_{1,p}$.	$-\frac{t_p}{\mu_p} K_{1,p}$.
$p=1$	1	0·177	0·177	−0·023
$p=2$	0·977	0·017	0·194	−0·156
$p=3$	0·821	−0·123	0·071	−0·004
$p=4$	0·817	−0·122	−0·051	+0·046
$p=5$	0·863	0	−0·051	+0·007
$p=6$	0·870	0·159	0·108	...

Similarly, by working through the system from the other end we have from equation (2) :—

——	$\frac{\partial K_{p,6}}{\partial \kappa_p}$.	$\kappa_p \frac{\partial K_{p,6}}{\partial K_p}$.	$K_{p,6}$.	$-\frac{t_{p-1}}{\mu_{p-1}} K_{p,6}$.
$p=6$	1	0·183	0·183	−0·024
$p=5$	0·976	0	0·183	−0·165
$p=4$	0·811	−0·122	0·061	−0·004
$p=3$	0·807	−0·121	−0·060	+0·048
$p=2$	0·855	+0·016	−0·044	+0·006
$p=1$	0·861	+0·152	0·108	...

Thus, the focal length is $\frac{1}{0\cdot108}=9\cdot26$ in., and the distances of the principal foci from the first and last surfaces are $\frac{0\cdot861}{0\cdot108}=7\cdot97$ in. and $\frac{0\cdot870}{0\cdot108}=8\cdot05$ in. respectively. Every figure is recorded above that is needed by a pupil using a slide rule. The whole calculation takes only a few minutes and the final agreement provides a good check on the accuracy of the figures. The case of telescopic systems could be dealt with equally simply.

In conclusion, he would refer to the use of the term "principal plane." He had been under the impression that the word principal was restricted to the cases when the first and last media were the same, so that the unit points and nodal points coincided. This, whether generally adopted or not at present, appeared to be a convenient usage, and it would be advantageous in the more general case to use the descriptive term "unit" plane rather than the more conventional term "principal."

Mr. W. R. Bower replies as follows:—The critics of the Paper on "Optical Imagery" have, as requested, paid attention to the value of the method to the student, but misconception has arisen concerning the meaning of "beginners." When speaking of these the author had in mind not so much those who are learning the alphabet of optical science, but somewhat more advanced pupils, as for instance, the students engaged in a higher course of physics who would take mathematics as a subsidiary subject. Such students would have proved the empirical rules of refraction at an earlier stage, and to these, perhaps, no harm is done by using, as a basis, the definition of μ as a velocity-ratio. However, teachers who object to this, could start by reversing § 3 (i). In the Paper as it stands there is no use for the sine law of refraction. Judging by the criticisms offered, it now seems important to say that the Paper and diagrams as published were not intended for students. The extent of the Paper would have been unduly increased if it had been arranged expressly for them. It is anticipated that a teacher will himself divide the diagrams and expand the text as he may think necessary. The student would then make careful drawings, by using mathematical instruments, of the cases considered. These would form an approximate sequel to laboratory experiment and a substitution for some of the problems of the exercise class. With many students undoubtedly an intelligible graphic method leads to a better understanding of a subject than that obtained by algebraic methods only. And in optics especially the value of well executed drawings, giving a true picture of the course of the rays, is possibly as great as that of the mechanical models that are educationally of so much advantage in the various branches of science. The method suggested in the Paper should practically have the value of a moving model. The graphic method of the Paper develops in a natural manner, and the usual algebraic formulæ may be derived with very little special artifice. It should therefore be intelligible and educational. The opinions of eminent teachers on the advantage of models are stated in Dr. S. P. Thompson's Presidential Address in Vol. XVII. of the "Proceedings" of this Society.

The author agrees with Mr. Smith that our present text-books make the subject more difficult and of much greater length than is necessary. Something will have been gained by this discussion if the thick lens obtains in the future a warmer welcome than it has received in the past. Cornu's junction-point, too, has so far had very scanty recognition in this country.

XXVII. *The Spectroscopic Resolution of an Arbitrary Function.* *By* C. V. BURTON, *D.Sc.*

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1. From time to time, since Dr. Schuster made known his conception of the periodogram, I have attempted to devise some quasi-spectroscopic method of analysing graphic records —not realising that the resolution could be affected by the spectroscope itself. Though I have not at present the means of working out the practical details of the method, a short communication on the subject may perhaps be justified by the simplicity and completeness of the optical theory, and the advantage which should result to more than one branch of science from a successful application of the principle proposed.

2. An ordinary grating has periodic rulings, and a spectrum obtained by means of it is characteristic of the radiation entering the spectroscope slit. On the other hand, if the radiation is homogeneous, while the distribution of the rulings is arbitrary, we obtain a spectrum characteristic of the *grating*. With reservations to be mentioned immediately, it should suffice if we can produce a transmission grating whose transparency (variable in one dimension only) is suitably related to the record in question. When this *equivalent grating* takes its place in a spectroscopic train, the slit being fed with homogeneous light, we shall be able to see or to photograph the spectrum of the record. The necessary reservations are: (i.) The effect of the transmission grating on any given element of the (normally) transmitted disturbance must be simply a reduction of amplitude, with no change of phase except such as is uniform over the whole extent of the grating,* and (ii.) the absorbing layer should be flat enough and thin enough to be sensibly equivalent to a plane surface whose power of absorption varies from point to point.

3. For convenience let the permeability (P) in the neighbourhood of any point be defined as the square root of the reciprocal of the "density," so that at normal incidence the

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* This will be at least very approximately the case if the partial opacity is due to a distribution of small perfectly opaque granules, with transparent interspaces through which the luminous disturbance can penetrate.

proportion of luminous energy transmitted is P^2. A simple harmonic grating* is defined by

$$P = A_0 + A_1 \cos px,$$

where A_0 and A_1 are constants of which A_0 is necessarily the greater, $2\pi/p$ is the grating interval and P is the permeability at every point of that vertical line whose horizontal co-ordinate is x. With the slit illuminated by homogeneous light of wavelength λ $(=2\pi/n$, say), and with the incidence normal, such a grating gives a central image of intensity proportional to A_0^2, and on each side a *single* spectral image, at an emergent angle θ given, in accordance with the ordinary grating formula, by

$$p = n \sin \theta. \qquad \text{(1)}$$

4. Assuming still the conditions postulated in § 2, consider the general case of a grating of width b extending from $x = -\tfrac{1}{2}b$ to $x = \tfrac{1}{2}b$. Let the permeability P be a function of x only, and equal to $P_0 + \varphi(x)$, where P_0 is a positive constant, while the mean value of $\varphi(x)$ is zero. Within the limits $x = \mp \tfrac{1}{2}b$, the value of $P_0 + \varphi(x)$ must evidently lie between 0 and 1; beyond those limits it vanishes. Let homogeneous light proceeding from a point source and rendered parallel by an object glass fall normally on the grating, and for simplicity suppose that this light is plane-polarised, the plane of polarisation being either parallel or perpendicular to the rulings. In either case the disturbance incident upon the grating may be specified by writing down the value of the electric vector as a function of the time. Let this be

$$A \cos nVt,$$

so that V is the velocity of light. The transmitted disturbance is accordingly

$$AP \cos nVt = AP_0 \cos nVt + A\varphi(x) \cos nVt.$$

5. The first term on the right gives rise to the central image and need not be further considered. To determine the effect of the remaining term, imagine a telescope of assigned focal length pointed at the centre of the grating in an azimuth inclined θ to the grating normal. The amplitude of the luminous disturbance reaching the focus of the telescope has to be expressed as a function of θ. Factors which are not variable with θ may be omitted; nor are we here concerned with variations in a vertical sense, since these in practice would

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* *Cf.* Rayleigh, "Wave Theory of Light," § 15.

be obliterated by substituting an illuminated slit for the ideal point source. Corresponding to the emergent angle θ, the disturbance proceeding from any point x of the grating suffers a relative retardation $x \sin \theta$, and, with an arbitrary change in the origin of time, the disturbance at the focus of the telescope is seen to be proportional to

$$\psi(\theta)\int_{-\infty}^{+\infty} \cos n(\mathrm{V}t - x \sin \theta)\varphi(x)\, dx, \quad . \quad . \quad . \quad (2)$$

in which the limits of integration might equally well have been given as $\mp\frac{1}{2}b$. The factor $\psi(\theta)$ in general differs from unity, because each element of the disturbance emerging in the plane of the grating is oblique to the direction of the beam considered. From symmetry $\psi(\theta)$ is an even function of θ, and in the ideal case, where the effect of the grating is entirely one of absorption, its form is simple: for an incident beam polarised perpendicularly to the rulings, $\psi(\theta)$ is constant and equal to unity; for polarisation parallel to the rulings, $\psi(\theta) = \cos \theta$. If the incident light is unpolarised (which implies that the homogeneousness is not mathematically absolute) the factor corresponding to $\psi(\theta)$ will be $\surd\{\frac{1}{2}(1+\cos^2 \theta)\}$, the emergent beam being perceptibly polarised except for very moderate values of θ.

6. In virtue of (1) the expression (2) may be written

$$\left.\begin{array}{l} \psi(\theta)\{f_1(p) \cos n\mathrm{V}t + f_2(p) \sin n\mathrm{V}t\}, \\ \text{where} \quad \begin{array}{c} f_1(p) \\ f_2(p) \end{array} = \displaystyle\int_{-\infty}^{+\infty} \begin{array}{c} \cos \\ \sin \end{array} pu\varphi(u)du \end{array}\right\} . \quad . \quad . \quad . \quad (3)$$

The corresponding intensity of illumination is

$$\mathrm{B}\{\psi(\theta)\}^2[\{f_1(p)\}^2 + \{f_2(p)\}^2], \quad . \quad . \quad . \quad . \quad (4)$$

where B is an instrumental constant. If the brightness has been mapped throughout the spectrum, and if the value of B and the form of the function $\psi(\theta)\}^2$ have been independently found, we shall be able to plot the values of $\{f_1(p)\}^2 + \{f_2(p)\}^2$ as ordinates against abscissæ proportional to p or to the grating interval $2\pi/p$.

7. Where only moderate values of θ are in question, so that θ^2 and higher powers may be neglected, $\psi(\theta)$ is sensibly a constant, and moreover the spectrum may be photographed on a flat plate. If the exposure law of the plate is known, the photograph may be regarded as equivalent to a periodogram,*

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* *See* footnote to § 11.

the case being so far analogous to that of an ordinary spectrogram. In the present instance, however, the scale of abscissæ is not one of wave-lengths, but of *reciprocal* grating intervals (p).

8. Now by Fourier's theorem

$$\varphi(x)=\frac{1}{\pi}\int_0^\infty f_1(p)\cos pxdp+\frac{1}{\pi}\int_0^\infty f_2(p)\sin pxdp, \quad . \quad . \quad (5)$$

$f_1(p)$, $f_2(p)$ having the values defined in (3), and a full determination of the functions f_1, f_2 would constitute a complete Fourier analysis of $\varphi(x)$.

If we put

$$\left.\begin{aligned}\frac{1}{\pi}[\{f_1(p)\}^2+\{f_2(p)\}^2]^{\frac{1}{2}}&=\mathrm{K},\\ f_1(p)/\pi\mathrm{K}&=\cos\chi,\\ f_2(p)/\pi\mathrm{K}&=\sin\chi,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(5) becomes

$$\varphi(x)=\int_0^\infty \mathrm{K}\cos(pt-\chi)dp, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and it is K^2 as a function of p which we are in a position to determine spectroscopically. We have yet to ascertain the phases of the various harmonic constituents, that is, to determine χ as a function of p. There are, broadly speaking, two cases in which the χ's have a definite significance. One is the case of a function $\varphi(x)$, whose effective values are confined to a limited range of value of the argument; χ is then in general a continuous function of p. The other is the case where $\varphi(x)$ is wholly or partly made up of absolutely pure harmonic constituents, corresponding to discrete values of p. The corresponding χ's are then precisely definite, and the greater the range over which our Fourier analysis is extended the more accurately they can be determined.

9. To find the χ's, but in each case with an ambiguity of half a period, we may proceed as follows: Imagine the grating bisected along a line perpendicular to the rulings, and one-half of it rotated through 180 deg. in its own plane, carrying with it its origin of x. There will then be a single vertical *line of agreement*, where like values of x come together. If now one-half of the grating is moved lengthwise micrometrically, any given line of the spectrum will show periodic variations of brightness, the minima being extinctions under proper conditions of adjustment. The minima will occur when the line of agreement is at a node of the harmonic grating constituent responsible for the line examined. The positions of the nodes having thus been determined, the ambiguity can in most cases be removed by integration over a relatively short portion of

the original graphic record. I think this plan might be quite useful in the case of a banded spectrum, derived from a function of limited range for, when the phase had been determined without ambiguity (say) for the dominant period of the band, the distribution of phase throughout the band could probably be followed without much difficulty. Perhaps the chief use of the spectroscopic method would be to indicate what periods are worth examining in more laborious ways.

10. Where the incident light is as nearly as possible homogeneous, and the periods of the grating are to be investigated, the spectroscopic conditions are in some respects unusual. The limited width of the grating still imposes a minimum interval which must exist between the centres of two lines if they are to be seen as separate; but, given the grating as the subject of observation, no irrelevant broadening of lines is here implied. The grating is in fact analysed into simple harmonic gratings of infinite width, and, in the absence of aberrations, with a sufficiently narrow slit, the performance is only limited by the value of $\lambda/\Delta\lambda$, where $\Delta\lambda$ is the "effective range" of wave-length comprised in the incident radiation. Now $\lambda/\Delta\lambda$ can easily be 400,000 or more, so that we can count on all the spectroscopic detail to which our record entitles us, though for photographing it upon a convenient scale a rather long camera may be needed.

11. For moderate angles of emergence, as already stated, we have a sensibly uniform scale of *reciprocal* grating intervals, the middle of the scale corresponding with $2\pi/p=\infty$. The original graph will in many cases represent a function $H\varphi(t)$ of the time t, H being in general a dimensional constant. In that case the spectrum seen will be disposed according to a sensibly uniform scale of *frequencies.*[*] The whole spectrum

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[*] The function $\phi(x)$ within the range $x=\mp b/2$ has throughout been treated as the entire subject of analysis. This would be the natural mode of proceeding when, for example, $H\phi(t)$ was the ordinate of a seismogram, whose outstanding features were confined to a correspondingly restricted space of time. $\{f_1(p)\}^2+\{f_2(p)\}^2$ is then characteristic of the function ϕ *as a whole*, and is a factor of what I propose to call the *integral periodogram*, a term which has Prof. Schuster's approval. On the other hand, as Schuster has pointed out, when the effective values of the function to be analysed extend with statistical uniformity over a wide range, of which only a relatively small *sample*, extending over a range b, is taken for analysis, the *average* value obtained for $\{f_1(p)\}^2+\{f_2(p)\}^2$ will be sensibly proportional to b, and on division by b will furnish quantities characteristic solely of the function in question. In all such cases $b^{-1}[\{f_1(p)\}^2+\{f_2(p)\}^2]$ is a factor of the ordinate of the periodogram, the term being now used in its accepted sense, without qualification. *Cf.* Schuster, "Proc. R.S.," A. 77, pp. 136–140 (1906).

being in light of one wave-length, the comparison of intensities, whether visually or photographically, is simplified. In particular the "density" in the neighbourhood of any point of the grating has a single definite value.

12. Remembering that it is the square root of the reciprocal of the photographic density whose values are to be (within a constant term) proportional to the ordinates of the record, we have first to determine the law connecting density with exposure, and in the apparatus which I have provisionally planned the problem of producing the equivalent grating from a given record then resolves itself into the suitable shaping of a cam. A good test of the performance would be to produce truly periodic gratings "equivalent" to pure sine curves of long and of short period, and to examine these in the spectroscope for harmonics. Various considerations indicate that a fairly open scale for the grating is desirable.

Note added May 20th, 1913.

The resolution of a transmission grating into harmonic constituents may be illustrated by a familiar example. Consider a grating consisting of a regular succession of narrow strips of greater and of lesser transparency alternately. The graph representing the distribution of permeability (§ 3) is in this case simply a "chess-board," and is readily analysable into a series of sine curves, each of which corresponds to an harmonic grating constituent. This compound character of the grating is evidenced by the existence of second and higher order spectra in addition to the spectrum of the first order. When the radiation examined by means of such a grating is sensibly homogeneous, a spectrum of any given order consists of a single fine line, and the aggregate of these lines on one side or the other of the central image is the spectrum of the grating.

Since the foregoing Paper was in type, I have had the advantage of discussing the matter with Prof. H. H. Turner, whom I have to thank for a number of valuable suggestions. As a result, the practical aspect of the method now appears more simple than I had thought possible, and I have begun making some preliminary tests.

Lord Rayleigh's Paper, referred to in his communicated remarks, seems to have been unaccountably overlooked by those interested in harmonic analysis. It was only a day or two before my own attention had been drawn to this Paper that I had realised the possibility of analysing a function by means of a grating whose rulings are of variable *length* instead of variable intensity. The grating is thus reducible to a template, whose outline is the graph of the function to be analysed; but certain instrumental restrictions are at the same time imposed. The (vertical) slit of the spectroscope must be shortened to a pinhole, and attention must be confined to a linear horizontal strip of field, the central line of which is the horizontal axis of the diffraction pattern. These are precisely the conditions indicated and experimentally verified by Lord Rayleigh, who refers, in the Paper already cited, to the inconvenience which they entail. But as regards adaptability to practical needs, Lord Rayleigh's method seems to me distinctly more promising than that described above, for it demands no knowledge of the exposure law of a photographic plate, thus avoiding more than one serious difficulty, while the special drawbacks which have been referred to can be overcome by comparatively simple devices.

ABSTRACT.

An ordinary grating has periodic rulings, and a spectrum obtained by means of it is characteristic of the radiation entering the spectroscope-slit. But if the radiation is homogeneous, while the distribution of the rulings is arbitrary, we obtain a spectrum characteristic of the grating. It is thus found to be theoretically possible to resolve spectroscopically a given arbitrary function $\phi(x)$ into its harmonic constituents. The "permeability" of a photographic negative at any point being defined as the square root of the reciprocal of the "density," the first step is to make an "equivalent grating." This is a plate whose permeability (variable in one dimension only) has at any point x the value $A+B\phi(x)$, where A and B are constants. When this (transmission) grating takes its place in a spectroscope whose slit is fed with homogeneous light, the spectrum of the function $\phi(x)$ can be seen or photographed. Suitably interpreted, it gives us the periodogram of $\phi(x)$. A device is described which it is hoped may prove useful for determining the phases of the various harmonic constituents.

The theory of the proposed method of resolving functions is discussed, and is as complete as that of ordinary spectroscopy, while in one respect it is more simple; for, since the light entering the spectroscope-slit is entirely of one wave-length, the comparison of intensities of spectral lines (whether visually or photographically) is facilitated.

Some preliminary practical tests are now being made.

DISCUSSION.

Lord RAYLEIGH communicated the following remarks: In connection with Dr. Burton's interesting Paper I should like to draw attention to somewhat similar suggestions that I made in 1903 ("Phil. Mag.," V., p. 238). It is to be wished that one or other of these optical methods should be applied to some practical problem of the analysis of an irregular curve.

The CHAIRMAN remarked that those who had gone through the labour of getting out a periodogram would appreciate any mechanical means of doing this. He understood that Dr. Burton considered Lord Rayleigh's method was better than his own.

The AUTHOR stated that he thought that Rayleigh's method had great advantages. In reply to Dr. A. Russell, he also stated that the phase of the harmonic components could be determined, but with an ambiguity of half a period. This ambiguity could be removed by a rough integration over a portion of the curve.

Mr. J. E. SEARS asked if all the periods given by this method of analysis were multiples of a fundamental as in the cases of the ordinary Fourier's analysis.

The AUTHOR stated that the present method gave the periodogram of a function as it was given by Fourier's double integral theorem. In reply to a further question from the same speaker, he stated that, in analysing a column of numerical values, the method would give all the detail that was actually involved in the figures themselves.

Mr. A. EAGLE (communicated remarks) thought it ought to be clearly pointed out that if it was desired to analyse a function extending over a limited range into the ordinary Fourier's Series, it would be necessary to repeat this function end to end a very large number of times in the grating. If it was only inserted once in the grating we should get the Fourier Integral analysis of a function, equal to the given function within the given limits, but zero everywhere outside these limits, which was quite a different thing. For any periodic function stretching from $-\infty$ to $+\infty$, it was easily seen that Fourier's Integral broke down into a Fourier's Series. Obviously, in this case, the periodogram must vanish for all wave-lengths that are not aliquot parts of the wave-length of the function analysed.

The AUTHOR entirely agrees with Mr. Eagle's remarks. So far he has not contemplated the application of his method to strictly periodic functions, because the coefficients of their various harmonic constituents can be so readily obtained by mechanical integration or otherwise.

XXVIII. *Some Experiments to detect β-Rays from Radium A.*

By W. MAKOWER, *M.A., D.Sc., and* S. RUSS, *D.Sc.*

RECEIVED MARCH 26, 1913. READ MAY 16, 1913.

WHEN an atom of radium A disintegrates, an α-particle is expelled which carries with it two positive atomic charges. At the same time, the radium B atom formed recoils with a single positive atomic charge. To account for these facts it is necessary to suppose that three negative electrons are expelled during the process. If these are emitted with a high velocity they should appear as β-rays capable of detection by the ionisation they produce or by their photographic action. On the other hand, they might consist of a slowly moving δ-radiation, which would escape detection by either of the above methods. The experiments described below, made in the hope of detecting β-rays from radium A, failed to reveal any such radiation. Experiments were first made to determine the rise of the β-radiation from a vessel filled with radium emanation carefully freed from the active deposit. If radium A emits β-rays, the radiation should increase much more rapidly from nothing than if no such radiation exists. A comparison of the initial stages of the rise curve with that to be expected if only radium B and radium C emit β-rays should, therefore, decide whether any β-rays are also emitted from radium A.

The experiment was carried out by exposing some radium emanation to a strong electric field for some hours to remove the active deposit formed, and then admitting the emanation through a plug of glass wool and through a tube immersed in liquid air into an evacuated space, one side of which was closed by a thin mica window. The thickness of the mica was equivalent to 4·3 cm. of air in its power of stopping α-rays. The mica window was placed opposite an aluminium window situated in the side of a β-ray electroscope; the two windows, together with the air between them, just sufficed to completely absorb the α-rays from radium C.

It has been shown by Fajans and Makower* that with a screen just thick enough to absorb the α-rays from radium C, the ratio of the ionisation produced by the β-rays from radium B to that produced by the β-rays from radium C is 1·33.

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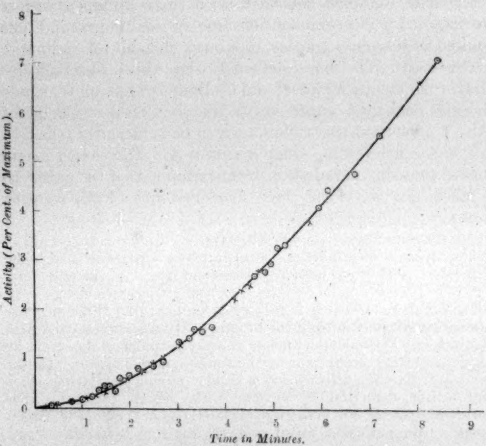
* Fajans and Makower, "Phil. Mag.," February, 1912.

Assuming this value, the variation with time of the ionisation produced by radium emanation freed from the active deposit can be calculated with the help of the table given by Moseley and Makower.* The curve calculated in this way is plotted in the figure. The experimental values obtained in two series of experiments are also shown, and will be seen to lie closely on the theoretical curve. The points in the two experiments have been made to fall on the theoretical curve at 8 minutes 2 seconds and 8 minutes 22 seconds respectively. This was done since it was found difficult to obtain with sufficient accuracy the whole curve from the start of the experiment until the maximum activity was attained. The agreement between experiment and theory leaves no doubt that under the experimental conditions no β-rays entered the electroscope. This is in agreement with some recent observations by Danysz.† There still, however, remains the possibility of the existence of β-rays too soft to penetrate the absorbing material between the emanation and the electroscope.



Rise of Activity from Pure Emanation (β-rays).

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[*] Moseley and Makower, "Phil. Mag.," February, 1912.
[†] Danysz, "Le Radium," January, 1913.

To decide whether any very soft β-radiation exists, experiments were made by one of us (W. M.), using a photographic method. A bare wire was exposed for five minutes to radium emanation so as to collect on its surface a deposit of radium A. The wire was then quickly mounted in vacuo in a magnetic field in front of a narrow slit, and the magnetic spectrum of the radiation passing through the slit was investigated. Although photographic records of β-radiation were obtained, these were traced to radiation from the small quantity of radium B formed from the radium A during the experiment; for the lines obtained coincided with those obtained with a wire exposed to the emanation for several hours and not mounted in the photographic apparatus until all the radium A had decayed. The lines detected were those observed by v. Bayer, Hahn and Meitner* and by Danysz,† having velocities 0·36, 0·41 and 0·63, expressed as fractions of the velocity of light. The method was thus shown to be sufficiently sensitive to detect any β-rays from radium A. The experiments indicate that any β-radiation from radium A, if it exists at all, must have a velocity less than one-tenth of the velocity of light.

ABSTRACT.

When an atom of radium A disintegrates an α-particle is expelled which carries with it two positive atomic charges. At the same time the radium B atom formed, recoils with a single positive charge. To account for these facts it is necessary to suppose that three negative electrons are expelled during the process. If these are emitted with a high velocity they should appear as β-rays capable of detection by the ionisation they produce or by their photographic action. On the other hand, they might consist of a slowly moving δ radiation which would escape detection by either of the above methods. The experiments, which were made by both methods in the hope of detecting β-rays from radium A, failed to reveal any such radiation.

DISCUSSION.

Prof. C. H. LEES asked if Dr. Russ thought that the explanation was that the radiations were so soft that they could not be detected.

Dr. RUSS thought that this was unlikely, as they would probably have been detected if present.

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* v. Baeyer, Hahn and Meitner, "Phys. Zeitschr.," 12, p. 1099, 1911.

† Danysz, *loc. cit.*

XXIX.—*Dust Figures.* *By* J. Robinson, *M.Sc., Ph.D.*

Received May 1, 1913. Read May 16, 1913.

The conditions for ripple formation between the nodes of a Kundt's tube were investigated by König over 20 years ago,* and he put forward a satisfactory theory to account for them. He first of all explained why the ripples are generally obtained only with such heavy powders as sand and emery powder and not with lycopodium. He showed that particles of very fine powder are carried along with a stream of air and hence soon collect at the nodes. Heavy powders are not appreciably carried along by streams of air, and in the Kundt's tube they can be treated practically as at rest in the streams. He then put forward the theory that the ripple formation is due to the known hydrodynamical forces between particles in a stream. Considering a pair of particles and supposing them spheres, there is a repulsion between them if the line joining their centres is parallel to the stream and an attraction if perpendicular to the stream. By analogy with the case of iron filings in a magnetic field he accounted for the formation of ripples.

It has been suggested† that these forces are not in themselves sufficient, and that some constraint must be introduced, such as, for instance one due to viscosity. In the present Paper it will be shown that no such constraint is necessary.

König's formula for the component of the force between two spheres of radius R parallel to the direction of flow is

$$F = -\frac{\pi}{2}\sigma R^6 \omega^2 \frac{\cos\theta\,(3-5\cos^2\theta)}{r_0^{\,4}}, \quad . \quad . \quad . \quad . \quad (1)$$

where σ is the density of the air,
r_0 is the distance apart of the centres,
ω is the velocity of the fluid,
θ is the angle between the line joining the centres and the direction of the flow.

As this force depends on ω^2 it was possible to explain the fact that the distance apart of two ripples varies with the distance from a node; ω^2 is greatest at the antinode, hence the force between any two particles is greatest there, other things being the same. By a direct application of this formula* it was shown that if a_{12} is the distance apart of two ripples at the antinode and $a_{r\,.\,r+1}$ the distance apart of two ripples at a distance k from the antinode, then

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* "Wied. Ann.," XLII., 1891, pp. 353, 549.

† Cook, "Phil. Mag.," 1902, May, p. 471.

$$\frac{a_{r\,.\,r+1}}{a_{1\,.\,2}}=\left(\frac{\omega_r}{\omega_1}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where ω_1 and ω_r are the velocities at these places. In the case of the Kundt's tube this formula becomes

$$\frac{a_{r\,.\,r+1}}{a_{1\,.\,2}}=\cos^{\frac{3}{2}}\frac{\pi}{2}\frac{k}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

where $2l$ is the distance between two nodes. This formula (3) was found to agree with actual measurements made on the ripples.

Another point which was shown to be in harmony with König's theory was that the distance apart of the ripples is a function of the intensity of the sound. The intensity was varied by stroking the rod as violently and as lightly as possible. It was found that the mean distance apart of two ripples is greater the greater the intensity of the sound.

The same characteristics have been shown to hold for the dust figures produced by exciting an electric spark either at the end of an open tube or at a hole at the centre of a horizontal plate. In the case of lycopodium powder on a plate it was found that the ripples get nearer together as their distance from the spark increases. This suggested at once that the figures are produced by the sound of the spark. The velocity of the air diminishes as we recede from the spark, and by analogy with the Kundt's tube figures this at once accounts for the ripples getting nearer together. The formula (2) was shown to agree well with actual measurements on some figures. The same formula was obtained independently by Marsh and Nottage† and also verified. It is not easy to find definitely how the intensity of sound diminishes along the surface of a plate. König‡ arranged a spark so that the intensity of the sound would fall off according to the inverse distance law. This was arranged by sending the disturbance between two parallel plates at a distance of 3 mm. or 4 mm. apart. König found that formulæ (2) gave quite a satisfactory explanation of the dependence of ripple pitch on distance from the spark.

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* Robinson, "Phil. Mag.," April, 1910, p. 476.

† "Phys. Zeit.," 1911, p. 439. "Proc." Phy. Soc. of London, XXVII., Pt. IV., 1911.

‡ "Phys. Zeit.," XII., 1911, p. 993.

Ripples produced in an open tube by a spark at one end also show characteristics which have been explained by König's theory. The ripple pitch depends on the intensity of the sound of the spark in a way required by that theory, and is not intimately bound up with the wave-length of the short waves emitted by the spark.

König's formula (1) for the force between two small spheres was deduced on the assumption that their distance apart is large compared with the radius of either of them. In the case of dust figures the small spheres are so close together that doubts may be entertained as to whether the formula really applies to this case. In deducing equations (2) and (3) the assumption was made that König's formula (1) holds for the particles very close together. The total force of one ripple on a particle at the centre of a neighbouring ripple was obtained by summing the forces of each separate particle on it. The close argeement of these formulæ with experiment shows that the assumption is not far from the truth. König* has recently devoted some attention to this point, and has come to the conclusion from some experiments he made that his formula (1) expresses very closely the force between two spheres, quite independently of whether they are far apart or near together. This fact enables us to place more confidence in the formulæ (2) and (3).

There is another point of resemblance between the Kundt's tube and the electric dust figures, and that is that when under the influence of the sound they stand up in sharp walls. This at once distinguishes them from the sand ripples on the sea shore. This wall formation can be readily seen with the Kundt's tube when a heavy powder is used. It is also noticeable with lycopodium if the rod is stroked lightly. With the electric dust figures one of two methods can be used to demonstrate the wall formation. They can either be examined by a microscope or projected on a screen. These walls are so sharp that it is desirable in making measurements on ripples to take instantaneous photographs of them whilst under the influence of the sound.

König's theory is quite capable of offering an explanation of this wall formation. We will first of all, however, devote our attention to find whether the hydrodynamical forces are sufficient to account for the formation of ripples from a uniform distribution of powder.

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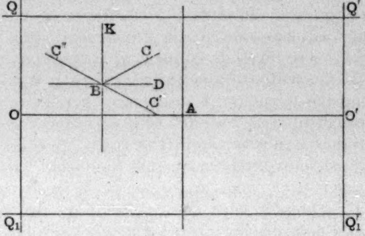
* "Phys. Zeit.," 1911, p. 991.

First consider the Kundt's tube. Let us suppose that the bottom of the Kundt's tube is plane and that sand is strewn uniformly over it. As soon as sound waves pass along the tube each particle of powder will be acted on by forces from all the other particles in the distribution. We will suppose that the tube is adjusted for resonance and restrict ourselves to the space between two nodes. Let QQ′ and Q'_1Q_1 represent the sides of the tube, their distance apart being $2b$ ($=2\cdot OQ$). Let OO′ represent the axis of the tube, the nodes being at O and O′ and the antinode at A. Between O and A the velocity of the air varies according to the equation



$$\omega=\omega_0\cos\frac{\pi}{2}\frac{x}{l},$$

where ω_0 s the velocity at the antinode A,

$2l$ is the distance apart of two nodes,

x is the distance measured from A.

König's formula (1) applies only to the case of ω constant and must be modified for application here. The necessary modification* is

$$F=-\frac{\pi}{2}\sigma R^6\omega\,\omega'\frac{\cos\theta(3-5\cos^2\theta)}{r_0^{\,4}}, \quad . \quad . \quad . \quad (4)$$

where ω and ω' are the velocities at the positions occupied by the two particles.

A particle at B is acted on by all the particles in the distribution immediately sound waves are started. The resultant

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[*] "Phil. Mag.," April, 1910, p. 476.

force on B would be zero if the distribution is uniform, the velocity the same throughout, and if B is far from the sides of the tube. For if we consider either the component of the force parallel to the axis or perpendicular to the axis, we can always find two particles symmetrically situated with regard to B whose attractions or repulsions on it balance each other. Considering the components of the forces which are attractive we must choose two particles C and C′ situated on opposite sides of the line BD which is parallel to the axis of the tube. For the repulsions we must choose the two particles C and C″ situated symmetrically with regard to a line BK perpendicular to the axis of the tube. We are thus unable to account for the ripple formation without removing the restriction that the velocity is uniform.

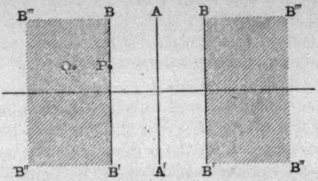
If we also remove the restriction that the point B is far from the sides of the tube we find that the attractions on it are not symmetrical unless it lies on the axis of the tube. There is thus an initial tendency for a particle at B to move towards the axis of the tube.

We will also find that there is a resultant force on this particle parallel to the axis of the tube if it is not at the antinode. For if B is some distance from the antinode, two particles symmetrically placed with regard to the line BK have not equal and opposite effects on B. As the velocity at C on the antinode side of B is greater than at C″ the repulsion of C on B is greater than that of C″ on B by equation (4). The same remark applies to all pairs of particles symmetrically placed with regard to BK. Hence there is a resultant force on a particle at B tending to move it to a node. Immediately sound waves are set up in the tube the uniform distribution of powder between two nodes is upset the particles which were initially at an antinode remain there, and all other particles are displaced in the direction towards a node. Thus we account for a ripple forming at the antinode.

This tendency of particles to a node was demonstrated by preparing a board of width just less than the diameter of the Kundt's tube. A ridge was made on it by making a fine saw cut across it and fastening in it a narrow strip of copper foil. The ridge projected ½ mm., 1 mm. and 2 mm. above the surface of the board on different occasions. Powder was strewn uniformly over the board, which was inserted in the tube. When the rod was stroked the powder jumped violently over the ridge, but always in the direction from an antinode towards a node. In this way sand grains could be made to jump nearly a centimetre above the surface of the board.

The ridges performed another function and showed that they take the place of ripples. A ridge acted just like a ripple in so far as there were ripples on both sides of it, and when under the action of the sound no powder was between the ridge and the neighbouring ripples. This function of a ripple of clearing the space between itself and a consecutive ripple of powder is characteristic, and can easily be observed whilst the rod is being stroked.

We have accounted for a ripple forming initially at the antinode. This ripple will repel particles near it, thus causing a clear space on each side of it. The distribution of powder is now somewhat as follows: At an antinode there is one ripple, AA′, on both sides of it a clear space, and then the particles are heaped together in some way at B, B′, B″, B‴. The width of the clear spaces AB will depend on the intensity of



the sound, for the forces exerted on particles by the antinode ripple depend on the intensity. Now this width AB fixes the distance apart of the ripples.

We can now readily account for the formation of the other ripples. A particle at P will be subjected to forces from the antinode ripple, and to those in the opposite direction from particles on the node side of it. These forces will be equal for a certain distance, AB, and then all particles at this distance will form into a second ripple. This in its turn will make a clear space on both sides, and so on. In this way we explain the initial formation of ripples purely on the fact of the hydrodynamical forces between two spheres in a moving fluid. It is unnecessary to postulate any other constraint on the particles, such as, for instance, friction.

Hence it is possible to obtain figures in the Kundt's tube without the assumption of any other forces whatsoever. In the case of the electric dust figures the velocity of the air along the tube or across the plate does not vary from zero to a maximum, *i.e.*, there are no nodes, and we must consider the whole distribution of powder. We can account for the formation of ripples in these cases by removing the restriction of a uniform distribution of powder, which is of course a very improbable distribution. A small discontinuity at any point causes the hydrodynamical forces to lose symmetry. By similar reasoning to the preceding for Kundt's tube this causes first one ripple and then others to form.

Once the ripples are formed we can consider how the particles in each ripple distribute themselves. Between any two particles in the tube there may be either an attraction or a repulsion. For two particles in the same ripple there is an attraction varying inversely as the fourth power of the distance between them. The particles of one ripple hence take up positions so that the mutual forces between them cannot make them move any more; in other words, the particles adjust themselves so that the line joining the centres of any two particles is perpendicular to the stream lines. Hence we see how the ripples are straight walls when under the action of the sound waves.

König's theory is thus found capable of explaining all known facts about the Kundt's tube and electric dust figures. The theory accounts as well for the necessary constraints for ripple formation as for the forces between the particles. Hence it is not necessary to look for constraints in the action of viscous forces, although, of course, the most rigid discussion of the figures must take these into account.

East London College, April 15, 1913.

ABSTRACT.

The ripple formation in Kundt's tube was first satisfactorily explained by W. König in 1891. His theory was based on the hydrodynamical forces between two particles in a stream. All the known facts about these figures fall into line with the theory. The distance apart of the ripples increases with the intensity of the sound, and also from the node to the antinode there is a variation of the ripple pitch as required by the theory.

Certain measurements on dust figures produced by an electric spark have shown that these figures also can be explained in a similar way to the Kundt's tube figures.

It was suggested by Cook some years ago that viscosity must be introduced in order to account for the formation of ripples. The

author shows that it is possible to account for ripple formation without introducing viscous forces at all. These will undoubtedly play some part, but more as a disturbance than as a help to the formation. In the case of the Kundt's tube figures there is a variation of velocity of the air from a node to an antinode, which produces a variation in the forces, and this causes the powder to lose its uniformity of distribution and to form ripples. The necessary constraints for the ripples are forthcoming without the introduction of viscosity.

DISCUSSION.

Prof. C. H. Lees thought that Dr. Robinson's Paper filled up a gap in the subject. He had been astonished to see the magnitude of the forces that seemed to act on the grains in some of the experiments Dr. Robinson had shown him.

Prof. J. W. Nicholson thought Dr. Robinson's explanation was extremely convincing. He would like to know what kind of accuracy had been obtained in verifying the formula for the variation of pitch of the fringes from node to antinode, as it was deduced on the assumption that the force between any two particles was uninfluenced by the remaining ones. This correction would probably not be large in practice. The assumption that viscosity was the important factor in the phenomena was obviously untenable.

Dr. A. Russell remarked that it seemed from the present Paper that the production of dust striations with an electric spark could not be regarded as proof that the discharge was oscillatory, as it often was.

Mr. F. E. Smith asked if Dr. Robinson had ever tried to get the ripples with a light powder such as lycopodium powder. Although the force diminished very rapidly with the radius of the particles, since, in practice, the distance apart would also diminish, the actual force per unit mass acting on the particles might be about the same.

The Author, in reply to Prof. Nicholson, stated that the accuracy with which the formula was verified so surprised König that he set some of his students to measure the force between two spheres. This formula held for distances apart that were comparable to the radius of the spheres. He did not think that the production of striations with an electric spark proved anything as to the oscillatory character of the discharge. In reply to Mr. F. E. Smith, he stated that it was quite possible to get the ripples with lycopodium powder if the intensity of the sound was kept very low. König investigated this point and showed that light powders were carried along and collected at the nodes. When this was prevented the ripples were obtained.

XXX. *Vibration Galvanometer Design.* *By* H. F. Haworth, *Ph.D., M.Sc., B.Eng., A.M.I.E.E.*

Received May 2, 1913. Read May 16, 1913.

In a previous Paper by the author ("Proc." Phys. Soc., Vol. XXIV., Part IV., June 15, 1912) it was shown that in a moving-coil vibration galvanometer the maximum sensibility as a voltage detector is obtained when the flux through the coil is so adjusted that at any time the back E.M.F. of the coil is equal to, and in phase with, its ohmic resistance drop (CR).

The losses in a vibration galvanometer may be divided into two parts, the electrical losses$=C^2R$, and mechanical losses, such as air and molecular friction.

The maximum deflection for a given applied voltage is produced when the electrical losses are equal to the mechanical losses, the efficiency of the instrument then being 50 per cent. In this case we may write mechanical losses$=C^2R$, so, for a constant applied voltage and resistance, the maximum amount of power capable of being used in producing motion is constant.

As the frequency increases the losses for a given deflection increase, and so the deflection for a given voltage will decrease, but in order to obtain the maximum mechanical output the back E.M.F. must be maintained constant.

If the losses with a given system vary directly as the frequency and deflection, then the deflection for a constant input will vary inversely as the frequency, but the back E.M.F. produced is proportional to the product of the frequency and deflection, *i.e.*, it will remain constant, which is the condition required. If the losses increase at a greater rate than the first power of the frequency, then, on account of the diminished deflection, the flux in the gap must be increased.

The vibration galvanometer might be used as a delicate testing machine in order to observe the behaviour of different metals under alternating forces of different values and frequencies. The mechanical input under the maximum sensibility conditions is exactly known, and air friction could be allowed for or eliminated by working in vacuum, and so the molecular losses under the given conditions could be determined.

If the mirror of the instrument had no mass the frequency would be inversely proportional to the length of wire. Now the back E.M.F. is proportional to L × B × *f*, where L is length of wire, B is flux density and *f* is frequency, *i.e.*, the back E.M.F. is proportional to B, L × *f* being constant, so alteration of length under the above circumstances would not necessitate an alteration of flux, but, owing to the mass of the mirror, the frequency increases at a slower rate than the inverse length, and, as the wire is shortened, the flux density must be increased to keep the back E.M.F. at its proper value.

If the instrument is used at its highest frequency the effective length of the wire is short compared with its total length (it may be only one-seventh), the losses are high and its sensibility is low. The mass of the mirror is now very important and it causes the working length of the wire to be much shorter than would be required if the mirror had no mass. The result is that at high frequencies the gap flux density must be high compared with that used for low frequencies.

The resistance of the wire of an ordinary Duddell vibration galvanometer is of the order of 140 ohms, and of this only about 20 ohms is effective at high frequencies, so the remaining 120 ohms may be considered as an added resistance. Not only must the flux density be increased on account of the previous reasons, but it must also be much larger than it need be on account of the resistance of the non-working part of the wire, in fact, the total increase of flux necessary for maximum sensibility at high frequencies is such that it is not easily obtainable with the present design.

Two improvements immediately suggest themselves :—

1. Cut out the non-working part of the wire, substituting for it wires of low resistance.
2. Shorten the length of the pole pieces.

The flux density required for long strings (9 cm. or 10 cm.) is about from 4,000 to 6,000 lines per square centimetre for phosphor bronze, and this may just as well be obtained by running a gap, say, 3 cm. deep at from 12,000 to 18,000 lines per square centimetre, as by having one of 9 cm. deep at 4,000 to 6,000 lines per square centimetre. The advantage comes in when the wire is shortened. If the total flux is kept constant the average working flux density of the wire increases as the length decreases until the ends of the wires are between the poles when the flux density remains constant.

The combination of the reduced resistance of the instrument with the increased flux density at the higher frequencies enables one, with the aid of a variable magnetic field, to obtain the maximum sensibility of the instrument over all its ranges with a more moderate size of electromagnet; in fact, for a wide frequency range, it would not be necessary to bring the bridge pieces between the pole pieces, thus allowing the length of the air-gap to be reduced and so making the necessary electromagnet still smaller.

The above alterations were carried out on the instrument previously used by the author; the original pole pieces were cut down from 10 cm. with square ends to 7 cm. with ends tapering to 4 cm. at the gap. Although the shape of the pole pieces did not lend itself to efficient working with a strong electromagnet, the old electromagnet, 7 cm. deep, used in the previous experiments being utilised, the increased flux density obtained was sufficient, in conjunction with the resistance alteration, to demonstrate the value of the improvements.

The superfluous wire was cut out at the top bridge by running the wire, after leaving the ivory bridge piece, under a small platinum-silver bridge fixed into the back part of the ivory bridge. The current was taken to the wires at the bottom bridge through two platinum-silver hooks rubbing on the wires. These hooks were soldered on to pieces of copper wire fixed into the ivory bridge; these pieces of copper wire projected on either side of the bridge piece each terminating in an eye which engaged on a thin stranded copper wire running parallel with the operating screw. These copper wires were carried at the top on small ivory studs fixed on the tube support at the middle of the galvanometer frame, and the bottom ends were soldered to lugs which also carried the working wires so that the non-working part of the wire and the copper flexible up to the bridge were always in parallel.

With the phosphor-bronze wire previously used the contacts made by the platinum-silver bridge pieces were not always good; the wire seemed to be patchy, so, as no new wire was at hand at the moment, a platinum-silver wire of the same diameter, 1·4 mils, was used. This was immediately successful, the resistance variation with length being very steady, the equation being $R=8+5{\cdot}7L$, where L equals length of string in centimetres.

The resistance of the portions of wire from the ivory bridge pieces up to the platinum-silver bridge pieces was about 8 ohms;

this is about one-third of the resistance of the galvanometer with the wires in the shortest position used (2·5 cm.=22 ohms), and this could be still further reduced if the bridge pieces were re-designed for this kind of work.

A series of experiments was carried out in which different string lengths and different tensions were used. In each case the flux was varied from a maximum to 10 per cent. of the maximum, and, for each value of the flux, the millimetres per micro-ampere and the millimetres per milli-volt at a metre were determined. From these results were obtained the apparent resistance of the instrument and the efficiency as determined by the millimetres per milli-micro-watt at a metre. The direct-current sensibility was also obtained and the resistance was measured.

The method of testing was similar to that described in the previous Paper, the only alteration being the substitution of a Paul unipivot dynamometer in series with R_1, which not only measured the current through R_3 but also the voltage across R_2. This gave a much quicker working arrangement. A portable Wheatstone Bridge was also introduced into the circuit in order to measure the galvanometer resistance. The results are summarised in the Table given; the first part of the results refer to the experiments previously made, using a phosphor-bronze wire, with some additional figures worked out, the wire being of constant resistance and working between pole pieces 10 cm. long. The platinum-silver wire readings were taken with the travelling contact bridges and the 4 cm. tapered pole pieces.

Comparing the phosphor-bronze and the platinum-silver wires under as near similar conditions as possible, taking the 9·5 cm. wires the following points will be noted :—

The maximum direct-current sensibility is greater in the case of the long string and long pole pieces than with long string and short pole pieces on account of the greater flux through the wire due to the lower reluctance of the circuit. As the string shortens the direct-current sensibilities decrease, but, owing to the fact that the flux density is constant in the first place, and is steadily increasing in the second case, with decrease of length until the bridge pieces enter the field, the falling off in sensibility takes place at a much greater rate with the long poles than with the short poles, and with a 6 cm. string the sensibilities are of the same order; after this the advantage lies with the short pole pieces.

Wire used, &c.	Length, cm.	Tension.	Frequency per sec.	Maximum voltmeter sensibility, mm.-milli-voltmetre.	Average flux density for this voltmetre sensibility.	Resistance of galvanometer.	Apparent resistance for max. sensibility.	Mm. per 10^{-9} watt at a metre for max. sensibility.	Max. mm. per micro-amp. at a metre for max. flux.	Mm. per 10^{-6} watt D.C. max. field.	D.C. sensibility, mm. per milli-amp. at a metre.	Current magnification due to resonance.
1·4 mil phosphor-bronze	9·5	1	237	95	4,630	137·5	275	26·0	46	159	148	310
		2	295	81	5,340			22·2	44	64	93·5	470
		3	320	74	5,940			20·3	38	38·3	72·5	525
Pole pieces 10 cm. long	6·5	1	310	73	8,270			20·0	26	41·7	75·7	343
		2	410	53	8,650			14·5	22	10·3	38·2	576
		3	526	41	9,280			11·3	14·5	3·7	22·6	640
				No maximum sensibility. Value at max, flux density.			varies up to					
	2·5	1	520	17	14,000		200	3·4	3	0·9	11·1	271
		2	740	14·4	14,000		200	2·6	2	0·23	5·6	358
		3	900	10	14,000		160	1·6	1·6	0·10	3·8	424
1·4 mil platinum-silver not annealed	10	1	237	84	2,740	64·8	139·6	11·7	26·0	198	113	230
		2	315	76				10·6	21·6	83	73·5	294
		3	360	69·6				9·7	20·0	44	53·3	375
	8	1	250	67·2	3,440	52·9	105·8	7·1	16·0	154	90·0	79
		2	392	68				7·2	15·2	39	45·5	158
		3	440	60·8				6·4	12·3	29	39·2	163
Pole pieces 4 cm. long	6	1	360	50·5	5,930	42·0	84	4·25	9·0	93	62·5	144
		2	494	46				3·87	8·7	22·5	30·8	282
		3	600	48·5				4·08	8·2	9·5	20 0	410
	4	1	495	44·5	10,000	31·5	63	2·80	5·3	26	28·6	185
		2	675	40				2·52	5·2	8·0	16·0	325
		3	860	36·7				2·31	3·0	3·9	11·1	270
	3	1	595	29	11,500	26·0	52	1·51	2·3	15·4	20·0	115
		2	800	29					1·85	5·0	11·8	157
		3	940	29					1·84	2·9	8·7	212
1·4 mil annealed Platinum-silver wire	9·5	1	237	98	2,260	63	126	12·3	28·4	170	103	276
		2	325	97				12·2	32·0	59·6	61·5	520
	2·5	1	740	37·5	11,550	23	46	1·73	2·0	6·4	12·1	165
		3	920	38·5				1·77	2·6	3·0	8·4	310

The same remarks apply to the figures for the alternating-current sensibility, though to a considerably smaller degree on account of the fact that we are now dealing with a vibrating system, and the losses in the phosphor bronze are apparently smaller at low frequencies and long strings than with platinum silver. At high frequencies with short strings the wires seem equally efficient; this is seen on examining the deflections per watt.

The mechanical losses for maximum sensibility are equal to $E \times I = \frac{V^2}{4R}$ or $\frac{V^2}{4} = R \times$ mechanical losses, so for a constant applied voltage the losses vary inversely as the resistance. Working at a point of maximum reception of energy for producing mechanical work, we have naturally a constant watt sensibility for a given deflection, because the losses for a given deflection and frequency are constant. If we increase the galvanometer resistance we must correspondingly increase the flux density to still keep at this point; the back E.M.F. must increase correspondingly with the resistance. The sensibility of the instrument as an ammeter is independent of the resistance, but its sensitiveness as a voltage detector varies inversely as the resistance. If the frequency of the moving coil be now increased, the power required to vibrate the mirror through the same angle as before is increased, but, as the maximum amount of power available for vibrating the system, considering a constant resistance instrument, is still the same, the deflection, therefore, is decreased.

The back E.M.F. has still to be the same, the frequency is greater, so the flux may have to change to keep to the point of maximum sensibility. In the case of the phosphor-bronze wires the flux increased with the tension, but, using platinum-silver wire, the flux was practically constant with varying tensions. From the above and the consideration of the efficiency column, there is an indication that the losses in platinum silver increase at a slower rate with increase of tension than the losses in phosphor bronze.

The great advantage of a variable resistance instrument is seen at the higher frequencies. At maximum sensibility the watts input to the 2·5 cm. string is about three times that of the 10 cm. string; this is as it should be because as the frequency increases the sensibility diminishes. The short pole pieces now show to advantage because we are still able to put the total flux through the moving system and use it instead of having it acting on the inoperative part of the wire. The efficiency of this particular system is now equal to the phosphor bronze, the current sensibilities are of the same order, but the voltmeter sensibility of the instrument is about three to four times greater with platinum silver than with phosphor bronze. With a frequency range of about 4 to 1 the voltmeter sensibilities have dropped in the case of the constant resistance instrument in the ratio of 10 to 1 and with the variable resistance instrument in the ratio of 2·5—3 to 1, which is a great improvement.

Owing to the lower initial resistance of the platinum-silver wire, its voltmeter sensibility at low frequencies, although its losses are greater, is equal to that of the phosphor-bronze wire, while at high frequencies it is much superior.

A few readings were taken with an annealed platinum-silver wire. The original wire was not quite straight and could not be straightened by increasing the string tension; this caused the mirror to point in various directions as the bridge pieces were moved, so another wire was taken and heated to red heat by a current of 0·3 ampere, while a small weight kept the wire stretched. This cured the mirror trouble and gave an improved moving system. It is interesting to note that the efficiency of the short tight platinum-silver wire is equal to that of the short phosphor-bronze wire.

The flux densities are average values over a field 2 mm. wide, so are only comparative and do not necessarily represent the flux density through the wire.

I have to thank Mr. F. W. Andrews of the City and Guilds (Engineering) College for his skill and ingenuity in making the necessary alterations.

ABSTRACT.

1. The maximum amount of power available for vibrating the moving system of a vibration galvanometer of the moving-coil type is $V^2/4R$. As the frequency of the instrument is raised the losses increase rapidly, so it is an advantage to be able to increase the useful power input per unit voltage. To do this the resistance of the instrument must be decreased. This can be done in a galvanometer of the Duddell type by leading the current in and out at the bottom bridge and short-circuiting the wires at the top bridge, and it results in a great increase of sensibility. A lower resistance also requires a lower flux density.

2. Owing to the losses in the moving system increasing at a greater rate than the first power of the frequency, and that the frequency of the system increases at a slower rate than the reciprocal length of the

string on account of the mass of the mirror, the flux density must be increased as the frequency increases in order that the back E.M.F. of the moving system may always be half that of the applied P.D. As the losses are low at low frequency and the mass of the mirror is not large, then, compared with the mass of the wire, the flux density required is moderate ; but at high frequencies the flux density required is large. In order to obtain this result economically it is convenient to make the depth of the poles small compared with the maximum length of the wires. This gives a sufficient field for the long wires, and for short wires one is able to obtain the necessary flux density because the total flux can still be put through the moving system.

3. A combination of 1 and 2 makes a very satisfactory instrument with a much flatter voltmeter-sensibility-frequency curve than obtained, usually.

DISCUSSION.

Mr. W. **Duddell** remarked that the Paper dealt chiefly with varying the length of the vibrating wires, but if it was required to make a really efficient galvanometer for high frequencies it was best to re-design the whole instrument, as the distance apart of the wires, the scale distance, size of mirror, &c., all ought to be altered. These various factors could be calculated beforehand, and the instrument relied upon for behaving according to the calculations. When a very low resistance was wanted silver wires could be used. He was rather afraid that sliding contacts would not be found satisfactory. It was quite true that a strong field in the centre of the wire was as good as a weak one for the whole length, but a strong field necessitated using an electromagnet, against which he had a prejudice, for vibration galvanometers. This necessitated a battery to excite it, and not only made the insulation more difficult, but also greatly increased the capacity to earth which for high frequencies it was very necessary to keep low. In his own laboratory, if he connected one terminal of a vibration galvanometer, tuned to the alternating electric light mains, to earth, he could get a very considerable deflection simply due to capacity currents. He was glad to see that Dr. Haworth had pointed out that the back E.M.F. of a vibration galvanometer behaved like a resistance since it was proportional to the applied E.M.F.

Mr. A. **Campbell** pointed out that it was not in all cases necessary that the instrument should have as great a voltage sensitivity as possible. In some cases it was current sensitivity that was required. This depended upon the bridge the galvanometer was used with. With respect to Dr. Haworth's suggestion that the vibration galvanometer could be used as a testing machine for studying the elastic behaviour of wires, he had thought the same himself for some time, but engineers replied that it was not the properties of wires that they wanted. These had become so altered during the process of drawing as to be useless to apply to other specimens. The tests would be useful as giving the properties of wires at high frequency. In this connection he drew attention to Kapp's machine for testing materials under rapidly alternating stresses. He asked if Dr. Haworth could give any information as to how the dynamical hysteresis varied with the angle of twist. He fancied it varied as a fairly high power. He believed that nearly all the hysteresis damping was due to the bending and not to the twisting of the wires. He had had the same experience as Mr. Duddell with respect to capacity currents.

Dr. A. **Russell** thought that a few more formulæ would have been desirable in the Paper. He remarked that the expression $V^2/4R$ for the energy absorbed was capable of very easy proof as follows : Taking i, e and

e_b as being the instantaneous values of the current, the applied voltage and the back E.M.F. respectively, we have $ie_b = \frac{e_b(e-e_b)}{R}$. Taking mean values this gave $W = \frac{VV_b \cos a}{R} - \frac{V_b^2}{R}$, where W was the power absorbed and V and V_b were the root mean square applied and back E.M.F.'s respectively, and a was the phase angle between them. This expression was very easily seen to give the maximum value of $\frac{V^2 \cos^2 a}{4R}$, when $V_b = V \cos a$. When adjusted for resonance $\cos a$ became unity.

The AUTHOR, in reply to Mr. Campbell, stated that he had not worked out the hysteresis law.

XXXI.—*Electrothermal Phenomena at the Contact of Two Conductors, with a Theory of a Class of Radiotelegraph Detectors.* *By* W. H. Eccles, *D.Sc.*

Received May 16, 1913. Read May 30, 1913.

When an electric current is caused to pass across the interface between a pair of conducting masses, heat is in general liberated or absorbed in accordance with the law of Peltier. When the masses are in contact over a very small area, as, for example, when a cylinder of graphite is laid across a copper wire, there may be, in addition, appreciable generation of heat in accordance with the law of Joule. If the substances constituting the contact are bad conductors of electricity and of heat, and if they stand far apart in the thermoelectric series, the phenomena arising when a current is forced across the joint become very striking, for in such circumstances relatively large amounts of heat may be developed, the heat is conserved, and therefore the thermoelectric effects enhanced.

It is evident that the thermoelectric forces called up by the local heating may assist or may oppose the E.M.F. applied to produce the current and that the phenomena of asymmetric conduction at once arises. But besides the Joule and Peltier effects, the Thomson effect may contribute to the phenomena. In the case of bad conductors of heat the temperature gradients very near the contact will be very steep, and thus the Thomson effect will be localised in the immediate neighbourhood of the contact.

Further than this it is obvious that, on account of the temperature changes, the portions of the substances near the contact will suffer a change in the magnitude of their electrical resistivity. It has been shown* that this effect alone leads to remarkable and important results, and is sufficient to account for all the principal features of the single-point coherer used in wireless telegraphy.

The thermoelectric forces and the changes of electrical resistance that arise from differences of temperature are much greater in combinations of such substances as iron pyrites than in combinations of ordinary metals. A pyrites-lead couple yields an E.M.F. some 200 per cent. greater than a bismuth-lead couple, between the same extremes of (ordinary) temperature; while the temperature coefficient of resistance of pyrites is about four times as great numerically as that of copper. But the thermal conductivity of pyrites is so very much smaller than that of lead that all these thermoelectric phenomena are greatly accentuated in the former case. Contacts between non-metallic conductors are of special interest for the reason that the bulk of the wireless telegraphy of the world is carried on by aid of detectors that consist of nothing else than a contact involving at least one non-metallic conductor.

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* "Proc." Physical Society of London, Vol. XXII., p. 869.

The subject does not appear to have been discussed hitherto from the theoretical standpoint adopted in this Paper.

Thermoelectric Theory of a Contact.

The thermo-electric constants of such substances as pyrites, zincite, carborundum, &c., are not easy to measure accurately and their coefficients of increase of resistance with temperature are exceedingly difficult to determine. The author has made numerous determinations, and has found that all the materials examined follow with fair precision the ordinary thermoelectric law that their thermo-electric powers are linear functions of the temperature, and also that their temperature-resistance coefficients are all large and negative.

For a thermocouple formed of (say) pyrite and lead we may therefore assume the E.M.F. equation

$$e_\tau = \alpha\tau^2 + \beta\tau + \text{constant}, \quad \ldots\ldots \quad (1)$$

where τ represents the absolute temperature of the hot junction, α and β are constants, and e_τ is the E.M.F. when one junction is at temperature τ and the other at the absolute zero. The ordinary thermodynamic theory may legitimately be applied. Thus if σ represent the specific heat of electricity in the substance, P the Peltier effect and p the thermoelectric power at the temperature τ, the first and second laws of thermodynamics give, when applied to a cycle of infinitesimal range,

$$\sigma = \frac{dP}{d\tau} - p,$$

and
$$\frac{d}{d\tau}\left(\frac{P}{\tau}\right) = \frac{\sigma}{\tau}.$$

Whence we have
$$\sigma = \tau\frac{dp}{d\tau} = \tau\frac{d^2e}{d\tau^2}.$$

If we represent the absolute temperature of the surroundings and of the cold junction by z, and the temperature of the hot junction by $\theta+z$, we find from these equations that at the temperature of the surroundings

$$\sigma=2az,\quad p=\beta+\sigma,\quad \mathrm{P}=pz,$$

so that it is easy to calculate the thermoelectric quantities from the experimental equation (1).

Further, we obtain from (1)

$$\begin{aligned} e_\theta &= a\theta^2+(\beta+2az)\theta \\ &= p\theta+\theta^2/2z \\ &= \theta(\mathrm{P}+\tfrac{1}{2}\sigma\theta)/z. \qquad (2) \end{aligned}$$

When instead of lead a substance is used of which the thermoelectric properties are represented by σ', p', P', the E.M.F. in the circuit, when the junctions are at temperatures z, $\theta+z$, is

$$e_\theta=\theta\{\mathrm{P}-\mathrm{P}'+\tfrac{1}{2}(\sigma-\sigma')\theta\}/z. \qquad (3)$$

Now, suppose the circuit made up of these two conductors

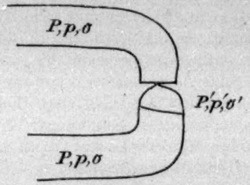

Fig. 1.

to have one junction forming a loose contact, the other a very good contact, as indicated in Fig. 1, where one conductor may be regarded as a piece of pyrites and the other as a piece of metal. Let an E.M.F. be applied to the circuit in any manner so as to produce a current in the direction opposite to the E.M.F. that would be produced by heating the contact. The heat liberated near the contact, say h, is, per absolute unit of electricity flowing across the contact, equal to the thermoelectric force e_θ plus the heat absorbed at the cold junction. That is

$$h=e_\theta+\mathrm{P}-\mathrm{P}'.$$

This heat tends to be concentrated in a small volume of the substance near the contact, but is dispersed continually by thermal conductivity and radiation. We will assume that the rate of loss of heat by these agencies is, as a whole, proportional to the excess of temperature of the junction over its surroundings. The rise of temperature causes an alteration of the electrical resistance of the joint; let the true resistance at any temperature θ above the temperature of the surroundings be expressed by $\rho(1+\gamma\theta)$, where γ is a temperature coefficient dependent on both the substances at the contact.

By equating the rate of accumulation of heat to the rate of liberation of heat by the electrical processes less the rate of loss of heat to the surroundings by purely thermal processes, we have, in mechanical units,

$$k\frac{d\theta}{dt}+m\theta=hy+\rho(1+\gamma\theta)y^2, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where y is the current in the direction defined as above, k is a constant involving the densities and specific heats of both substances and m is a constant involving the internal and external thermal conductivities of the substances near the contact.

Let the part of the circuit which suffers inappreciable temperature change possess the constant resistance r and inductance L, and let the applied E.M.F. be e, at any time t, then

$$\mathrm{L}\frac{dy}{dt}+(r+\rho)y+e_\theta=e. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We have to deal in this Paper with a state of affairs in which the rate of variation of current and applied E.M.F. is relatively slow. In this case we may use the equations of the steady state, namely,

$$m\theta=hy+\rho(1+\gamma \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$(r+\rho)y=e-e_\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substitute for e_θ from equation (3) above, eliminate θ, and we obtain an equation for the current y sent across the junction by the applied E.M.F. e. The equation is less cumbrous, however, if we obtain it in terms of the P.D. between the extreme parts of the conductors which vary in temperature. Regarding this as a portion of the applied E.M.F. and representing it by x, we have

$$x=c-ry+\mathrm{P}-\mathrm{P}'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Then the current equation becomes

$$\frac{\sigma-\sigma'}{2m^2z}x^2y^2+\frac{\rho\gamma}{m}xy^2+\frac{\mathrm{P}-\mathrm{P}'}{mz}xy-x+\rho y+\mathrm{P}-\mathrm{P}'=0, \quad . \quad (9)$$

which may be written

$$ax^2y^2+cxy^2+bxy-x+\rho y+b'=0. \quad . \quad . \quad . \quad (10)$$

The constant coefficients a, c, b in this equation are, it will be seen, mainly dependent on the Thomson effect, the change of resistance with temperature, and the Peltier effect respectively. The term b' is always negligible except very near the origin of co-ordinates and will be put zero in all that follows. Subject to the proper interpretation of the variable x, the equation is that of the "steady current characteristic" of the combination of conductors. And since when b' is taken zero

$$x=e-ry, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the steady current characteristic, as usually drawn from observations of applied E.M.F. and consequent current, is identically that obtained by applying to the curve drawn from the above equation a homogeneous shear of amount r parallel to the x axis.

Particular Cases.

Case A,—$a=b=0$.

The coefficient a is proportional to the difference between the Thomson effects in the two conductors at the absolute temperature z; if they are equal it vanishes. The coefficient b is proportional to the Peltier E.M.F. at the junction at temperature z; it vanishes if the neutral temperature of the substances happen to be z. For thermoelectric effects to be wholly absent a and b must both be zero. We are then left with the cubic

$$cxy^2-x+\rho y=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which has been discussed in Papers* by the present writer. The quantity c depends mainly on γ, the coefficient of increase of resistance with rise of temperature. The most interesting cases arise when γ is negative, as it nearly always is in contacts including a native crystalline oxide or sulphide, or carborundum. In fact, this equation explains the action of the single-point coherer fairly perfectly; and thus, when in equation (10) this quantity c is finite and negative, we may say that this "coherer action" is accompanying the thermoelectric effects.

The kinds of curve that may be yielded by pure coherer action are shown in Figs. 2 and 3. In Fig. 2 the equation is

$$c'xy^2+x-\rho y=0,$$

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* "On Coherers," "Proc. Phys. Soc.," Vol. XXII. "On an Oscillation Detector,' "Proc. Phy. Soc.", Vol. XXII.

when c' is put for $-c$. The curve of Fig. 2 has only one real asymptote, the axis of y. In order to exhibit the curve connecting applied E.M.F. and resulting current the curve must be submitted to shear of amount r, where r is the non-varying

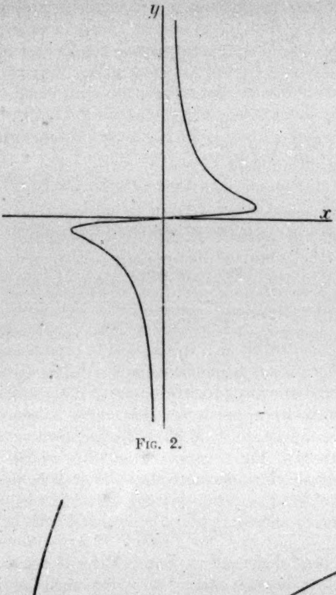

FIG. 2.

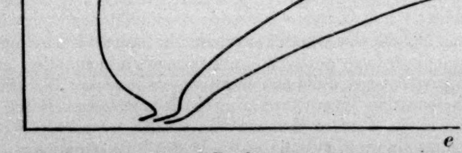

FIG 3.

part of the resistance of the whole of the circuit containing the contact. In this way we obtain curves of the kind shown in Fig. 3, which resemble those obtained[*] from a contact consisting of two pieces of galena. There are two vertical tangents if $8r<\rho$ (case I.) and none if $8r>\rho$ (case III.). The two vertical tangents coincide if $8r=\rho$ (case II.). Here ρ is the starting value of the variable part of the true resistance of the contact.

The curve I. shows the existence of instability; near the right-hand vertical tangent a very small increase of E.M.F. will produce discontinuously a great increase in the current; case III. is quite stable and illustrates a "self-restoring" coherer.

For contrast, the curve corresponding to the case when resistance increases with rise of temperature is shown in Fig. 4. The equation is

$$cxy^2-x+\rho y=0.$$

The part of immediate physical interest is that passing through the origin. It shows that, as the current increases, greater and greater increments of E.M.F. are needed to produce an assigned increase of current, as might be concluded without calculation. This is the kind of curve obtained in any circuit in which a conductor with positive resistance coefficient becomes heated; for example, a fuse wire raised towards fusion in air. The curve of case III., Fig. 3, arises when the heated conductor possesses a negative coefficient—*e.g.*, the carbon filament of a lamp.

The contact or aggregate of contacts that occurs between the carbon brushes and the commutator of a dynamo affords a striking instance of the phenomena. Many of the known properties of brush contacts have been explained by W. H. F. Murdoch [†] on these lines.

Case B.—$a=0$, $c=0$.

When the specific heats of electricity in the two conductors are equal $a=0$, and when there is no coherer action $c=0$. In this case the equation reduces to

$$bxy-x+\rho y=0, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

which gives the hyperbola shown in Fig. 5. Only the branch through the origin appears to be physically relevant. When sheared the equation is

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[*] "On an Oscillation Detector," *loc. cit.*

[†] Murdoch, "Commutator Resistance and Energy Losses," "Electrica Review," June 16, 1911.

$$bry^2-bey+e-(r+\rho)y=0.$$

This represents a hyperbola with its centre at the point $(r-\rho)/b$, $1/b$), and with asymptotes $by=1$, $b(e-ry)+\rho=0$; it is shown in Fig. 6. It is clear that the curve is not symmetrical about the origin, and, therefore, that the existence of the Peltier effect makes the contact a rectifier of alternating current of any amplitude.

Case C.—No coherer action. $\gamma=0$, and, therefore, $c=0$.

In this case the full effects of thermoelectric action, free from coherer action, are brought out. The equation is

$$ax^2y^2+bxy-x+\rho=0. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Two sub-cases arise—namely, a positive and a negative.

1. *a Positive.*—This sub-case is shown in Fig. 7. The branch through the origin shows that the contact will act as an excellent rectifier, unless, indeed, the invariable part r of the resistance is relatively very large; for even after considerable shearing the curve will have its relevant negative portion much steeper than the positive portion. The contrast with the last case (Thomson effect zero) is especially noteworthy. In that case the difference of gradient on the two sides of the axis of y, after shearing, is not nearly so pronounced as in the present case.

2. *a Negative.*—When a is negative we obtain the curve shown in Fig. 8. It has its steepest part in the positive quadrant, and the points of inflexion near the origin indicate that the rectification is more pronounced than in the last case. Thus we may expect a combination in which a is of opposite sign to b to behave as a good detector for the purposes of wireless telegraphy, other things being not unsuitable. The meaning of this condition for good rectifying properties may easily be expressed by aid of the diagrams of thermoelectric power in Figs. 9 and 10.

In both figures the Peltier E.M.F.s P, P′ divided by the absolute temperature z are represented by zp and zp'; and the specific heats of electricity σ, σ' divided by the temperature z are represented by the gradients of the lines at the temperature z. Now

$$a=\frac{\sigma-\sigma'}{2m^2z} \quad \text{and} \quad b=\frac{\mathrm{P}-\mathrm{P}'}{mz}.$$

Hence in Fig. 9 both a and b are positive, while in Fig. 10 a is negative and b remains positive. In the one case the neutral temperature is below z, in the other above z. This is to be

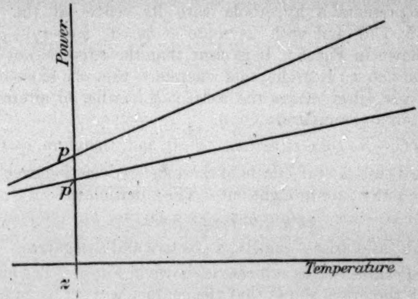

FIG. 9.

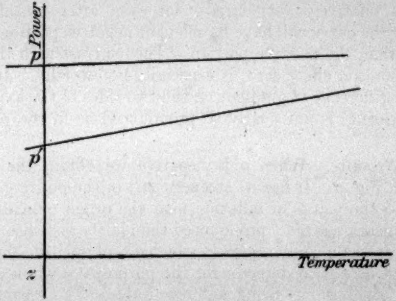

FIG. 10.

seen also from the familiar fact that the neutral temperature T is given by

$$T = -\frac{P - P'}{\sigma - \sigma'},$$

which in our notation is written

$$T = -\frac{b}{2ma}.$$

In words the theorem now arrived at is that the best rectifying contact obtained by purely thermoelectric effects is that in which the neutral temperature (or "temperature of inversion") for the pair of substances forming the contact is above the temperature of the cold junction.

Case D.—$c=0$ and $b=0$.

The above conclusion suggests an inquiry as to the rectifying properties to be expected of a contact composed of two substances which have the same Peltier E.M.F. with respect to lead and still have no coherer action. The equation is now, c being still zero,

$$ax^2y^2-x+\rho y=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

Two sub-cases arise, a positive and a negative.

1. *a Positive.*—The curve is given in Fig. 11. It indicates good rectification; a *negative* E.M.F. causes a greater flow of current than a positive E.M.F.

2. *a Negative.*—The curve is given in Fig. 12. In this case again there is good rectification, but now a *positive* E.M.F. causes the greater flow of current.

The principal difference between this case and the case C2 is that there is no point of inflexion near the origin in the present case. This again indicates that it is advantageous to have the neutral temperature high. In fact, referring back to case C, when b is not zero, and solving the corresponding equation as a quadratic in y, we have

$$y=\frac{-(bx+\rho)\pm\sqrt{(bx+\rho)^2+4ax^3}}{2ax^2}.$$

The vertical tangent to the curve is, therefore, given by the cubic equation for x

$$-4ax^3=(bx+\rho)^2.$$

First, taking a positive and of fixed value we see that the one real root of this equation is numerically greater the greater b, being $-(\rho/4a)^{\frac{1}{3}}$ in the limiting case $b=0$. Thus the greater the value of b the further does the vertical part of the physical branch of the curve depart from the axis of y. Since the neutral temperature is proportional to b, we may say that the rectification* produced by such a contact is greater the higher the neutral temperature. This is, moreover, the more marked the smaller the value of ρ. Now, taking a negative we see that the real positive root of the equation is greater the greater b, and thus again the rectification is greater the higher the neutral temperature. Further, supposing that in turn b is fixed in value, the form of the equation shows at once that the smaller the numerical value of a the greater numerically is the relevant root in x. Thus again it appears that a high neutral temperature corresponds to good rectification. But in all these conclusions there is always to be kept in mind the practical condition that the root of the cubic shall be a value of x that is attainable experimentally.

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* *See* Appendix p. 286.

Taken as a whole the argument establishes the theorem that the rectifying properties of a contact at which there is no coherer action are the more pronounced the higher (within practical limits) the temperature of inversion.

Case E.—The Resistance Coefficient not Zero.

The most interesting sub-cases occur when the Thomson effect in the circuit is zero. The steady current equation is

$$cxy^2+bxy-x+\rho y=0. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

There are three sub-cases :—

1. c Positive—that is, the resistance of the substances near the contact on the whole increases with rise of temperature. There are two real asymptotes parallel to the axis of x and on opposite sides of that axis. The curve (Fig. 13), is unsymmetrical about the origin and the contact possesses rectifying properties to some extent, but the asymmetry is smaller the greater c.

2. c Negative and $16c^2<b^4$. Here coherer action enters, but only to a relatively small extent. The two real asymptotes parallel to the axis of x are now on the negative side of that axis, and consequently the curve (Fig. 14) is very unsymmetrical about the origin. The contact will be a good rectifier, and will be a good detector of oscillations when submitted to a suitable positive E.M.F.

3. c Negative and $16c^2>b^4$. Here coherer action is predominant. The curve is shown in Fig. 15. The asymptotes parallel to the axis of x are imaginary, and the contact, when employed as a detector, is sensitive under a particular negative as well as a particular positive E.M.F. Its difference from a pure coherer would be seen experimentally by a difference in the numerical values of the particular positive and negative E.M.F.s that give maximum sensitiveness. The vertical tangents occur at values of x given by

$$\rho/(2\sqrt{-c}+b) \text{ and } -\rho/(2\sqrt{-c}-b).$$

GENERAL CASE.

We have now to analyse the general case when none of the constants in the steady current curve vanish. From what has gone before it may be expected that, broadly, the effect of the thermoelectric terms is to produce asymmetry and the effect of the resistance terms is to encourage symmetry in the physical portions of the curves.

The general equation is

$$ax^2y^2+cxy^2+bxy-x+\rho y=0. \quad . \quad . \quad . \quad . \quad (17)$$

By definition b is positive, and the positive direction of the current is that in which it must flow in order to absorb heat by the Peltier effect at the junction that remains at the temperature of the surroundings. The resistance ρ is essentially positive; the Thomson term a may be negative or positive, and c has the same sign as the aggregate resistance temperature coefficient of the contact.

There is no need to discuss here the method of plotting the curves, nor is there any necessity for plotting the curves accurately. Therefore in the figures only the general shape of each curve is intended to be conveyed, and though the non-physical branches have been inserted, isolated ovals have all been omitted. Asymptotes are shown by dotted straight lines. It is convenient to divide the discussion into cases delimited by the signs of the constants a and c. The reader may be reminded that the curves ought to be sheared in the manner already described before being compared with experimental curves.

Case I.—a Positive, c Positive.—The curves take the forms shown in Figs. 16A, 16B, 16C, according as b/ρ is less than, equal to, or greater than a/c, *i.e.*, $2\gamma(\mathrm{P}-\mathrm{P}')$ less than, equal to, or greater than $\sigma-\sigma'$. The greatest asymmetry is indicated in Fig. 16A, which corresponds to the condition that the Thomson coefficient shall be great compared with that of the Peltier effect and of the temperature coefficient of increase of resistance.

Case II.—a Positive, c Negative.—The existence of coherer action brings about two distinct sub-cases which both indicate that the contact would operate well as a detector for radiotelegraphy. In one of the cases the contact would be a sensitive detector only when the E.M.F. is negative; in the other it would be sensitive at two unequal and opposite values of the applied E.M.F. The analytical condition separating these cases is cumbrous (Figs. 17A and 17B).

Many samples of galena form with lead contacts of these types.

Case III.—a Negative, c Positive.—There is only one type of curve. A contact belonging to this class will behave as a sensitive detector only for positive applied E.M.F. (Fig. 18).

Case IV.—a Negative, c Negative.—There are four sub-cases shown in Figs. 19A, 19B, 19C, 19D. In A, b/ρ is much smaller than a/c—that is to say, $2\gamma(P-P') << \sigma-\sigma'$. In B, $2\gamma(P-P')$ is slightly smaller than $\sigma-\sigma'$. A case scarcely different from the last in its physical branch is given in *c*, where $2\gamma(P-P')=\sigma-\sigma'$. Finally, in D is seen the case when $2\gamma(P-P')>\sigma-\sigma'$. If we regard the passage through these sub-cases as being due to a gradual increase of the share of coherer action in the whole phenomenon, it is evident that coherer action always tends to make the realisable part of the curve symmetrical about the origin of co-ordinates. If, on the other hand, we regard the transition as due to gradual elevation of the neutral temperature, we see cases A to D correspond to higher and higher values of this temperature.

Contacts including zincite usually fall in one of these sub-cases.

Most of the curves drawn in this Paper possess a vertical tangent on one side of the origin of co-ordinates, and many of them a vertical tangent at each side. After allowing for the shearing process required to convert the curves into "characteristic curves," it may be concluded broadly that when a is positive the current produced by a definite voltage is greater when the voltage is negative than when it is positive; while, when a is negative or zero the positive voltage produces the greater current. Now it is well known that when a contact is warmed by direct communication of heat, the thermoelectric force is for some pairs of substances in *same* direction as the larger, and for other pairs in the *opposite* direction to the larger of the currents produced across the contact by reversing a constant applied E.M.F. The latter case has frequently been quoted as showing that these contacts did not owe their rectifying properties to thermoelectric effects. This is a fallacy. For the observation of the direction of the thermoelectric force produced by direct heating often shows only the sign of the Peltier effect and not the sign of the Thomson effect in the circuit; and it is the latter alone which decides, through the constant a of this Paper, the accordance or disaccordance indicated above.

Considerations relative to the Determination of the Constants involved in the Theory.

The constants occurring in the general steady current equation are a, b, c and ρ. By aid of the equations of the section dealing with the thermoelectric theory, two of these constants may be expressed partially in terms of the more fundamental constants appearing in the thermo E.M.F. equation

$$\begin{aligned} e_\theta &= \theta(\beta + 2\alpha z + \alpha\theta) \\ &= \theta(\beta' + \alpha\theta), \text{ say.} \end{aligned}$$

Here θ is the temperature of the contact above that of the cold junction and the surroundings z. We thus obtain

$$a = \frac{\alpha}{m^2}, \quad b = \frac{\beta'}{m}.$$

The quantity m is the measure of the rate of loss of heat by radiation and conduction.

Now, for any particular two substances forming a contact the constants α and β' may be determined by measurement of the thermoelectric force for two values of θ, and thus a and b become known if m can be determined. Unfortunately this quantity cannot be determined. Of the other constants in the current equation—namely c and ρ—the first is given by the equation

$$c = \rho\gamma/m.$$

The coefficient γ is, for the crystals that form the most interesting contacts, very difficult to measure. In all the methods the author has employed, difficulty has arisen through the high thermoelectric effect at the electrodes; but so far as the measurements go it is negative in many conducting native oxides and sulphides. The quantity ρ can be roughly estimated by shearing an experimental curve till its asymptote is parallel to the axis of y and then noticing that the tangent at the origin is $x - \rho y = 0$.

The data accumulated up to the present by the author will be published in a later communication, when also the experimental work of other observers will be discussed.

APPENDIX.

On the Operation of a Contact as a Radio-telegraphic Detector.

It appears that the earliest statement of the general connection between the steady current curve of a contact and its pro-

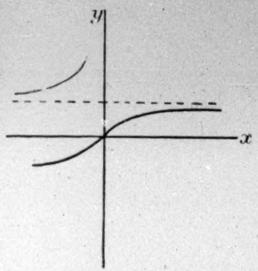

FIG. 4.

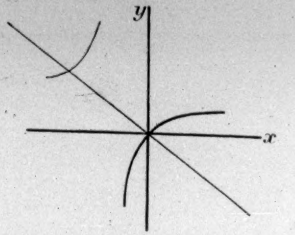

FIG. 5.

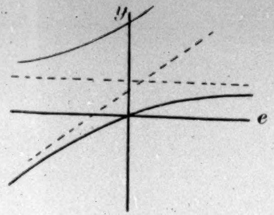

FIG. 6.

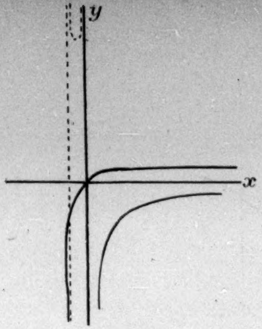

FIG. 16A.

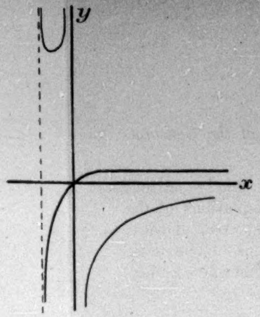

FIG. 16B

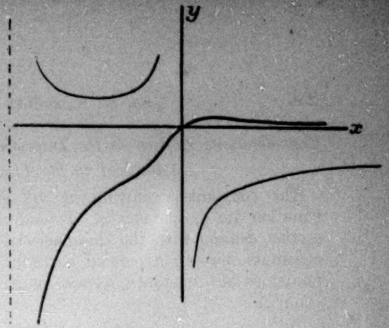

FIG. 16C

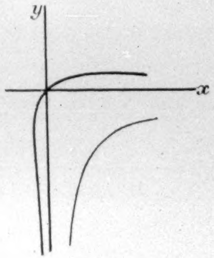

FIG. 7.

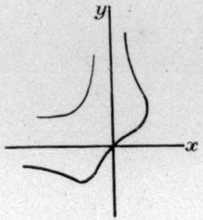

FIG. 8.

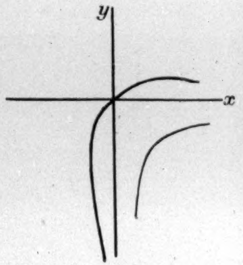

FIG. 11.

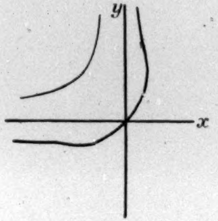

FIG. 12.

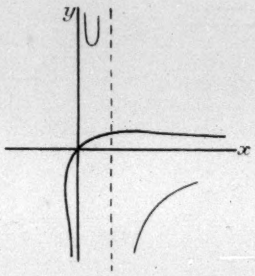

FIG. 17A.

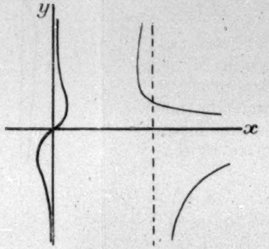

FIG. 17B.

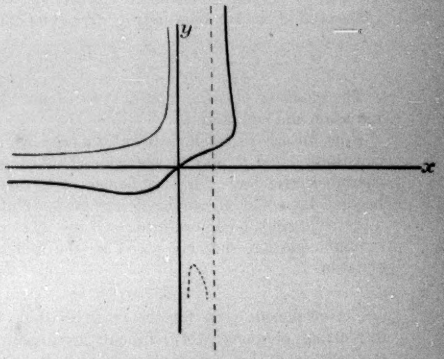

FIG. 18.

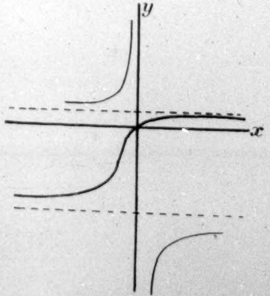

FIG. 13.

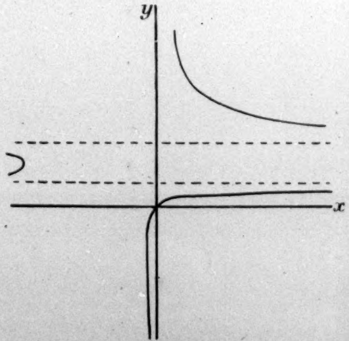

FIG. 14.

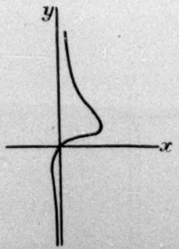

FIG. 15.

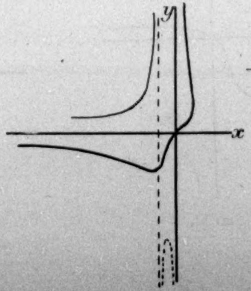

FIG. 19A.

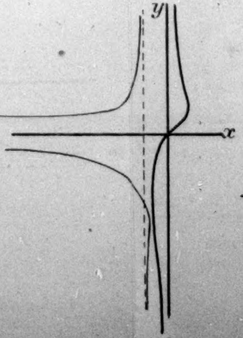

FIG. 19B.

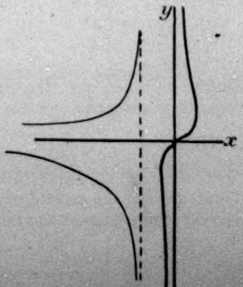

FIG. 19C.

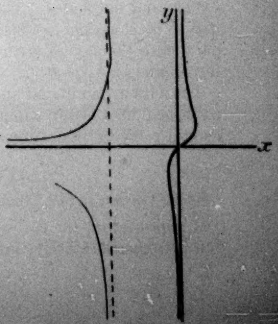

FIG. 19D.

To face p. 286.]

perty of acting as a detector of oscillations was made by H. Brandes* in 1906. He pointed out that any kind of conductor —not necessarily a contact—which departs from Ohm's law will rectify alternating current in some degree. If with such a substance (or contact) the volt-ampere curve is symmetrical in the first and third quadrants, a direct current of suitable magnitude must be kept flowing and the oscillatory current to be rectified must be superposed on this current. On the other hand, he pointed out, if the characteristic curve is not symmetrical about the origin, no direct E.M.F. is actually needed. And he showed that the same general rules hold good for all sorts of detectors, such as the electrolytic, as well as for the so-called crystal rectifiers. Again, Pickard† has indicated that the greatest sensitiveness of crystal contacts is at the E.M.F. where the rate of change of resistance is greatest, and J. A. Fleming‡ has published diagrams showing that in the case of his vacuum valve the E.M.F. of maximum sensitiveness is near that of the maximum ordinate of the second derived curve of the characteristic. These principles may now be regarded as well known and have been tacitly assumed in the preceding pages. It would be well, however, to examine the matter from the purely theoretical standpoint of this Paper.

Let the equation of the characteristic of any detector (Fig. 20) be $y=f(e)$, where y is the current produced by applying the steady E.M.F. e to the detector. Suppose the E.M.F. to increase by the small amount h and the current to increase in consequence by the amount k. Then, by Taylor's theorem,

$$k=hf'(e)+\frac{h^2}{\lfloor 2}f''(e)+\frac{h^3}{\lfloor 3}f'''(e)+,$$

where f', f'', f''' ... indicate the first, second, third ... derived functions of $f(e)$. We will suppose that h is a very small alternating E.M.F., say, $h=a \sin \omega t$, where ω is small.

The whole matter may now be regarded from two points of view. In the first place we may concentrate attention on the rectifying action of the contact, and calculate the excess of electricity passing in one direction round the circuit over that passing in the other direction, when the alternating E.M.F. is acting. In the second place we may endeavour to estimate, not what may be called the rectified current, but, instead, the energy possessed by the portion of the current in the circuit due to the harmonic E.M.F.

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* H. Brandes, "E.T.Z.," 27, p. 1015, Nov. 1, 1906.

† G. W. Pickard, "El World," November, 1906.

‡ J. A. Fleming, Royal Institution Discourse, June 4, 1909, and British Patent Specification 13518, 1908.

In the former case the average value of the increased current flowing in the circuit while the alternating E.M.F. is acting is

$$\delta y = \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} k\,dt,$$

where T is the period of the alternation and $\mathrm{T}=2\pi/\omega$. Hence

$$\delta y = \frac{a^2}{2\underline{|2}} f''(e) + \frac{3a^4}{8\underline{|4}} f^{\mathrm{iv}}(e) + \ldots .$$

Thus the magnitude of the average rate of flow of the rectified electricity depends on the derived functions of even orders as might be expected. The equation shows that if the steadily applied E.M.F. e is adjusted to a value corresponding to a sharp bend on the characteristic curve, then the rectified current may be very large, according to the value of $f''(e)$ principally. It is on this account that a coherer with a perfectly symmetrical characteristic can be made to act as a rectifier of slow alternating current.

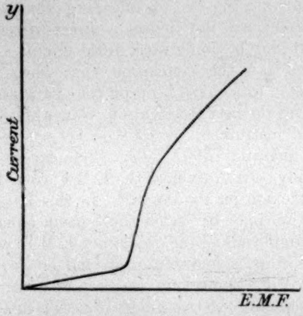

FIG. 20.

On the other hand, from the energy point of view, the increase of the mean square value of the current round the circuit, when alternating E.M.F. is imposed on the steady E.M.F., must be calculated. This increase is given by

$$(y+k)^2-y^2=\left\{f(e)+hf'(e)+\frac{h^2}{\underline{|2}}\cdot f''(e)+\right\}^2-\{f(e)\}^2$$

$$=2hf(e)\cdot f'(e)+h^2[\{f'(e)\}^2+f(e)\cdot f''(e)]$$

$$+h^3[\tfrac{1}{3}f(e)f'''(e)+f'(e)\cdot f''(e)+]+\ldots.$$

Thus, the additional work done in the circuit per second, in consequence of the imposition of the harmonic E.M.F. h, is a function of the gradient of the characteristic as well as of the higher derived functions. In particular, work will be done in the circuit even when using a straight portion of the characteristic curve where all the derived functions of even order might be zero, notwithstanding that in this case the rectified current would be zero. The apparent paradox that work may be done in the telephone circuit although there is no rectified current is removed by reflecting that a sine wave alternating current will not deflect a direct-current galvanometer though it may be doing work in the circuit.

We therefore draw the conclusion that when the steady E.M.F. is adjusted to a sharp bend on the characteristic curve the detector gives the best rectification; but when the greatest deflection of an instrument of hot-wire type is required, we must have the quantity $\{f'(e)\}^2+f(e)\cdot f''(e)$ near its maximum, which will usually occur very near the E.M.F. corresponding to the point of inflexion on the curve.* In this latter case the mode of operation may be styled the "mean square mode." In general the amplitude of the oscillatory E.M.F. will perhaps not be so minute as is here assumed, and consequently the local steady E.M.F. applied to the detector cannot be so adjusted as to separate the two modes of action just described.

The above analysis leads immediately to an expression for the efficiency of transformation of energy. The instantaneous rate of working of the applied oscillatory E.M.F. is hy, and thus, if W be the average rate of working,

$$W=\frac{1}{T}\int_0^T hydt$$

$$=\frac{a^2}{2}f'(e)+\frac{a^4}{16}f'''(e)+.$$

---

* In the case of the magnetic detector it has been shown experimentally by the author that the point of maximum sensitiveness is near, but not quite coincident with, the point of inflexion of the hysteresis loop.—"Proc. Phys. Soc.," Vol. XX., June, 1906.

The average rate of working in the circuit of the detector is

$$w=m\,.\,\frac{a^2}{2}[\{f'(e)\}^2+f(e)\,.\,f''(e)]+m\,.\,\frac{a^4}{32}[3\{f''(e)\}^2 \\ +4f'(e)\,.\,f'''(e)+f(e)\,.\,f^{\text{IV}}(e)]+\ldots,$$

where m is a proportional constant. The efficiency is the ratio $w : \mathrm{W}$.

The degree of the equation connecting w and W when a is eliminated between the above equations depends on the number of terms in powers of a^2 included in the process. The simplest case occurs when w is near its maximum and also

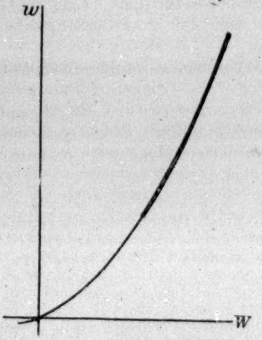

Fig. 21.

$f'''(e)=0$; for near the maximum of w the coefficient of a^4 in the equation for w vanishes, and then we have

$$wf'(e)=m[\{f'(e)\}^2+f(e)f''(e)]\mathrm{W}.$$

When W and w are taken as abscissæ and ordinates this equation connecting input and output of energy is represented by a straight line through the origin. More generally, writing the equation for w in the form

$$w=\alpha_1 a^2+\alpha_2 a^4$$

and

$$\mathrm{W}=\tfrac{1}{2}f'(e)a^2+\tfrac{1}{16}f'''(e)\,.\,a^4,$$

we have

$$\{\alpha_2\mathrm{W}-\tfrac{1}{16}f'''(e)w\}^2=\{\tfrac{1}{2}f'(e)\,.\,\alpha_2-\tfrac{1}{16}f'''(e)\,.\,\alpha_1\}\times\{\tfrac{1}{2}f'(e)w-\alpha_1\mathrm{W}\},$$

which represents a parabolic locus.

In the cases in which $f'''(e)$ and higher derived functions may be neglected—and these are often practical cases—this equation becomes of the form

$$4a_2(W+A)^2=f'(e)(w+B),$$

where A and B are positive constants involving $f(e)$, $f'(e)$ and $f''(e)$. The locus is shown in Fig. 21. The thick portion is that with which radio-telegraphy is concerned, for elsewhere W is negative or it is so small as to give no audible sound. Experimental observations made by the author on the magnetic detector,* the coherer,† the electrolytic detector and various contact rectifiers† have yielded points lying approximately on a straight line passing to the right of the origin in the manner suggested by a straight line drawn through the thickened part of the last figure.

From the experimental results mentioned, it would appear that this theory of the "detecting" property of any apparatus with a characteristic curve like Fig. 20, under low-frequency E.M.F. of small amplitude, holds good in some measure for the high frequencies and large amplitudes employed by the various observers.

ABSTRACT.

The Paper is a purely theoretical one, and deduces mathematically the laws connecting the current and the applied E.M.F. in a circuit containing a light contact of two conductors. When an electric current passes across a light contact of two different substances, heat is liberated or absorbed in accordance with the law of Peltier, heat is generated in accordance with the law of Joule, and, in the regions of the conductors where there is a temperature gradient, heat is liberated or absorbed in accordance with the laws of the Thomson effect. These thermal actions are very noticeable in contacts made of badly conducting natural oxides or sulphides on account of the high resistivity and the large thermoelectric effects in these substances. The low thermal conductivities of these substances exalt the electrical consequences by conserving the heat. The bulk of the wireless telegraphy of the world is carried on by such contacts as these, and the present Paper, therefore, constitutes a theory of the action of these detectors.

The general equation connecting the current across a contact and the P.D. between two points of the circuit is deduced, and is found to be of fourth degree when the difference of the specific heats of electricity in the conductors is not zero. The curves are discussed and plotted for all the principal variations of value of the Thomson effects, the Peltier effects, and the resistivity-temperature coefficients of the substances. It is shown that, in general, when a contact is used as a radiotelegraph detector, there may be some coherer

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* "Proc. Phys. Soc.," Vol. XX., 1906.

† Vol. XXII., 1910.

action mixed up with the thermoelectric "rectifying" action; and that, whether there is or is not any coherer action, the principal features of the characteristic curve connecting current and E.M.F. are determined by the Thomson rather than by the Peltier effect. Thus, if two contacts could be made having all their electrical and thermal properties of equal measure excepting the Thomson property, of which the measures were of opposite signs, the curve connecting current and E.M.F. for one contact would rise more rapidly on, say, the side of positive E.M.F. than on the side of negative E.M.F., and vice versa for the other contact; and this although a direct experiment of applying heat to the contacts in turn and observing the consequent thermoelectric forces, showed these to have the same direction in both.

DISCUSSION.

Dr. A. **Russell** considered Dr. Eccles' fundamental equation very valuable. He thought Dr. Eccles' definition of the efficiency of a detector useful and the method of arriving at it very neat, but he did not think the curve of efficiency given by Dr. Eccles would be much use. What was wanted to tell a good detector was the shape of the function showing the relation between the current and the E.M.F.

Prof. C. H. **Lees** thought it was important to have shown that the well-known phenomena of thermoelectricity and variation of resistance could explain the behaviour of contacts. The theory was quite elastic enough to explain more complicated characteristic curves than those now considered, as the resistance was not strictly a linear function of the temperature and the heat conductivity also was not a constant.

Mr. P. R. **Coursey** referred to the tests he had recently carried out for Dr. Fleming on the subject of crystal detectors, some of the results of which were before the meeting, and said they were undertaken more with the view to finding the relation (if any) that exists between the characteristic and sensitiveness curves for the crystals (taking impressed voltages as abcissæ in both cases) than to discovering a reason for the shape of the characteristic curves and the more or less abrupt changes of curvature, or "kinks," that occur in them, as had been done by Dr. Eccles in his Paper. It had been known for some time past that when the second differential of the characteristic curve of a Fleming oscillation valve is plotted it bears a considerable resemblance to the sensitiveness curve of the valve. This relation was found in the above tests to hold good to a greater or less degree in the majority of crystal detectors, the most notable exception being the galena-plumbago combination, for which practically no agreement between the two curves could be found. Similar remarks applied to the electrolytic detector which was also tested; and hence one was led to the conclusion that in most detectors we have at least two actions taking place tending to produce sounds in the telephones, and that these may or may not assist one another, it being probable that in the galena detector the effect due to the changes of curvature are almost entirely masked by some other, and opposing, effect due to a different cause—perhaps some type of electrolytic or thermal action. Hence to analyse or explain the shape of the characteristic curve of a detector in the manner outlined by Dr. Eccles was not necessarily going to be of assistance in finding a good detector, since, as seen from the above tests, the sensitiveness is not always dependent merely upon the shape of the characteristic curve, although nevertheless in many cases it will doubtless lead to valuable results.

Dr. J. A. **Fleming** communicated the following: With reference to Dr. Eccles' remarks on the characteristic curves of certain rectifying detectors, I believe I was the first to point out, in a British patent specification, No. 13,518 of 1908, the conditions under which certain rectifiers of oscillations, such as my glow-lamp detector, can be operated with an added E.M.F. as a more sensitive detector than by use as a mere rectifier without such E.M.F. Long before that date a large number of measurements of characteristic curves of my valves and other detectors had shown that there were sudden changes of curvature in them. The method of detecting these was to plot a first and second differential curve from the volt-ampere characteristic. In the above specification I described simple means of applying the boosting voltage in the case of a glow-lamp detector by using a potentiometer wire in connection with the same battery which incandesces the glow lamp. With certain detectors this is a more sensitive arrangement for detecting oscillations, but it is unfortunately as yet impossible to construct glow-lamp detectors or oscillation valves with predetermined characteristics. I have, however, one glow-lamp detector which when worked in this manner is more sensitive than any electric wave detector I have ever met. We have not yet found the way to reproduce it. I referred to these experiments in a Friday evening discourse at the Royal Institution, and exhibited a diagram showing the characteristic curve of a certain valve and its first and second differential curves. The scheme of the circuits and the method of working is fully explained with diagrams on p. 481 of Chapter VI. of the second edition of my book on "The Principles of Electric Wave Telegraphy and Telephony." Since that date a very large amount of information has been collected in my laboratory on the form of the characteristic curves of a number of contact and crystal detectors. Mr. Coursey made for me a very extensive collection of such measurements about eight or nine months ago, but this information and a numerous set of curves depicting the observations has never yet been published. The results do not, however, fall in with any very simple theory of such rectifying detectors. The point to be explained is the reason for this sudden change in curvature in the characteristic. It is obvious that if there is such a change of curvature then the mean value of the current through the detector and, therefore, in a telephone, in series with it, will be increased if we add to the alternating E.M.F. acting on the detector a steady unidirectional E.M.F. of value exactly corresponding to the abscissa of the characteristic at that point of change of curvature.

The Author, in reply to Prof. Lees, stated that all the coefficients of his fundamental equation were variable with temperature. In reply to Prof. Fleming, he stated that the Paper fully explains the main changes of curvature in the characteristic of a crystal detector, but does not touch on the theory of the vacuum valve.

XXXII.—*On the Evaluation of Certain Combinations of the Ber, Bei and Allied Functions.* *By* S. BUTTERWORTH, *M.Sc., Assistant Lecturer in Physics, School of Technology, Manchester.*

RECEIVED MAY 8, 1913. READ MAY 30, 1913.

1. It has been pointed out by Russell* that, in most formulæ, the ber and bei functions nearly always occur in one or other of the following combinations :—

$$\begin{aligned} X\ (x) &= \mathrm{ber}^2 x + \mathrm{bei}^2 x, \\ V\ (x) &= \mathrm{ber}'^2 x + \mathrm{bei}'^2 x, \\ Z\ (x) &= \mathrm{ber}\, x\ \mathrm{ber}' x + \mathrm{bei}\, x\ \mathrm{bei}' x, \\ W\ (x) &= \mathrm{ber}\, x\ \mathrm{bei}' x - \mathrm{bei}\, x\ \mathrm{ber}' x, \end{aligned}$$

while in most cases only the combinations Z/X, W/X, Z/V, W/V occur.

Russell has given formulæ for evaluating these combinations, suitable either for high or low values of the argument. These formulæ have been extended by Savidge† and he has published a four-figure table of the functions. A five-figure table of the combinations Z/V and W/V for smaller intervals of the argument has also been prepared by Rosa and Grover.‡

In the present Paper formulæ are established for functions φ_n and ψ_n defined by the relation

$$\varphi_n + \iota\psi_n = \frac{J_{n+1}(\sqrt{-\iota x})}{J_{n-1}(\sqrt{-\iota x})}.$$

From the definition of the ber and bei functions it may readily be shown that

$$\left. \begin{aligned} W/X &= x(1+\varphi_1)/2, \quad W/V = 2/x - x\psi_2/4 \\ Z/X &= -x\psi_1/2, \qquad Z/V = x(1+\varphi_2)/4 \end{aligned} \right\}, \quad . \quad . \quad (1)$$

so that Russell's formulæ are particular cases of the formulæ of this Paper.

The functions φ_n and ψ_n also occur in the problem of a spherical conductor in an alternating magnetic field, n in this case being of the form $(2m+1)/2$, where m is an integer.

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* Russell, "Phil. Mag.," 6, 17, p. 524, 1909.

† Savidge, "Phil. Mag.," 6, 19, p. 49, 1910.

‡ Rosa and Grover, "Bulletin" Bureau of Standards, p. 226, Jan., 1912.

2. *Formulæ for Low Values of x.*—By writing down the series for $J_{+}(x)$ and $J_{n-1}(x)$, and dividing, afterwards putting $\sqrt{-\iota}x$ for x, we get

$$\left.\begin{aligned}\varphi_n&=-\frac{2x^4}{(2n)^2(2n+2)(2n+4)}+\frac{(28n+48)x^8}{(2n)^4(2n+2)^2(2n+4)(2n+6)(2n+8)}\\ \psi_n&=-\frac{x^2}{2n(2n+2)}+\frac{(10n+12)x^6}{(2n)^3(2n+2)^2(2n+4)(2n+6)}\end{aligned}\right\}(A)$$

These formulæ are useful only for very small values of x.

3. *Formulæ for High Values of x.*—By making use of the relation

$$J_n(\iota x)=\iota^n I_n(x),$$

and using the semi-convergent expansion

$$I_n(x)=\frac{1}{\sqrt{2\pi x}}e_x\left\{1-\frac{4n^2-1}{8x}+\frac{(4n^2-1)(4n^2-9)}{2(8x)^2}-\dots\right\},$$

we get

$$\left.\begin{aligned}\varphi_n&=-1+\frac{2n}{\sqrt{2}\,.\,x}-\frac{2n(2n-1)(2n-3)}{8\sqrt{2}x^3}\\ &\qquad-\frac{2n(2n-1)(2n-3)}{8x^4}+\frac{2n(2n-1)(4n^2-9)(2n-7)}{\sqrt{2}\,.\,128x^5}\\ \psi_n&=-\frac{2n}{\sqrt{2}x}+\frac{2n(2n-1)}{2x^2}-\frac{2n(2n-1)(2n-3)}{8\sqrt{2}x^3}\\ &\qquad-\frac{2n(2n-1)(4n^2-9)(2n-7)}{\sqrt{2}\,.\,128x^5}\end{aligned}\right\}\quad(B)$$

Putting $n=1$ or 2, and making use of relation (1), Russell's formulæ for W/X, W/V, Z/X, Z/V are readily obtained.

4. *Formulæ for Intermediate Values of x.*—It is found on application that formulæ (A) and (B) fail for a considerable range of the argument.

If, however, the roots of $J_{n-1}(x)=0$ are known, this gap may be filled up.

Let $a_1, a_2, a_3 \dots$ be the positive roots of $J_{n-1}(x)=0$.

Then we may write

$$J_{n-1}(x)=a_n(x^2-a_1^2)(x^2-a_2^2)(x^2-a_3^2)\dots.$$

Since the series for $J_{n+1}(x)$ is more convergent than that for

$J_{n-1}(x)$, $J_{n+1}(x)/J_{n-1}(x)$ may be separated into partial fractions by the usual method, and we find

$$\frac{J_{n+1}}{J_{n-1}}=4nx^2\sum_{r=1}^{\infty}\frac{1}{a_r^2(a_r^2-x^2)}. \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Putting $\sqrt{-\iota x}$ for x,

$$\left.\begin{aligned}\varphi_n&=-4nx^4\sum_{r=1}\frac{1}{a_r^2(a_r^4+x^4)}\\ \psi_n&=-4nx^2\sum_{r=1}^{\infty}\frac{1}{a_r^4+x^4}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

These formulæ are always convergent, but the convergence is very slow.

However, by expanding the later terms in powers of x, practical formulæ may be obtained.

For values of x up to $x=a_2$, I find the most convenient form to use is

$$\left.\begin{aligned}\varphi_n&=-4nx^4\left\{\frac{1}{a_1^2(a_1^4+x^4)}+\frac{1}{a_2^4(a_2^4+x^4)}+C_3\right.\\ &\qquad\qquad\qquad\qquad\left.-C_5x^4+C_7x^8-\ldots\right\}\\ \psi_n&=-4nx^2\left\{\frac{1}{a_1^4+x^4}+\frac{1}{a_2^4+x^4}+C_2-C_4x^4+C_6x^8-\ldots\right\}\end{aligned}\right\}, \quad (C)$$

where

$$C_s=\sum_{r=3}^{\infty}\frac{1}{a_r^{2s}},$$

When $x>a_2$ formulæ (B) are suitable.

The coefficients C_s are easily determined by summation, except in the case of C_2. However, by comparing the coefficients of x^2 in formulæ (A) and (C) we find

$$C_2+\frac{1}{a_1^4}+\frac{1}{a_2^4}=\frac{1}{8n^2(2n+2)},$$

and this relation is sufficiently accurate to determine C_2.

5. *Numerical Values from Formulæ* (C).—The zeroes of $J_0(x)$ and $J_1(x)$ are given to a high degree of accuracy in Gray and Mathews' Bessel Function (pp. 244 and 280). From these values I obtain the values of the coefficients given in Table I. In Table II. are given the values of W/X, Z/X (*a*) as calculated by formulæ (C), using relation (1), (*b*) as calculated by Savidge.

The values of the argument chosen are those for which the calculation by the usual formulæ is most tedious.

In Table III. are the corresponding values of W/V and Z/V as compared with Rosa and Grover's values.

Numerical values from formulæ (B) are unnecessary, as these reduce to Russell's formulæ when $n=1$ or 2.

TABLE I.

s.	C_s (for $n=1$).	C_s (for $n=2$).
2	$2{\cdot}7338\times10^{-4}$	$1{\cdot}56456\times10^{-4}$
3	$2{\cdot}8933\times10^{-6}$	$1{\cdot}16285\times10^{-6}$
4	$3{\cdot}5003\times10^{-8}$	$9{\cdot}9744\times10^{-9}$
5	$4{\cdot}4599\times10^{-10}$	$9{\cdot}0709\times10^{-11}$
6	$5{\cdot}8173\times10^{-12}$	$8{\cdot}4836\times10^{-13}$
7	$7{\cdot}6745\times10^{-14}$	$8{\cdot}0504\times10^{-15}$
8	$1{\cdot}0183\times10^{-15}$	$7{\cdot}7129\times10^{-1}$
9	$1{\cdot}3552\times10^{-1}$	$7{\cdot}3960\times10^{-19}$
10	$1{\cdot}8065\times10^{-19}$	$7{\cdot}1222\times10^{-21}$
a_1	2·4048256	3·8317060
a_2	5·5200781	7·0155867

TABLE II.

x	W/X.		Z/X.	
	Savidge.	Formula (C).	Savidge.	Formula (C).
3	0·7485	0·748531	0·5399	0·539948
4	0·7141	0·714078	0·5842	0·584194
5	0·7101	0·710145	0·6040	0·603998
6	0·7101	0·710139	0·6211	0·621090

TABLE III.

x	W/V.		Z/V.	
	Rosa and Grover.	Formula (C).	Rosa and Grover.	Formula (C).
5	0·81709	0·817090	0·69496	0·694960
6	0·79786	0·797862	0·69781	0·697809
7	0·78377	0·783768	0·70037	0·700372
8	0·77361	0·773610	0·70214	0·702130

ABSTRACT.

Formulæ are obtained for the real and imaginary parts of J_{n+1}/J_{n-1}, when the argument is of the form $\sqrt{-\iota}\,x$. These formulæ are useful for evaluating the functions W/X, Z/X, W/V, Z/V defined by Russell, and have other applications.

The formulæ are of two types :—

(*a*) Those suitable for low or moderate values of the argument. These are particularly useful in those cases where computation by other methods is most laborious.

(*b*) Those suitable for high values of the argument. Russell's formluæ are particular cases of these formulæ.

XXXIII. *The Extraordinary Ray resulting from the Internal Reflection of an Extraordinary Ray at the Surface of an Uniaxal Crystal.* *By* JAMES WALKER, *M.A., Oxford.*

RECEIVED MAY 8TH, 1913. READ MAY 30, 1913.

TAKING one of the co-ordinate axes in the direction of the optic axis of the crystal, the equation of the extraordinary wave-surface may be written

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $b=c$, or $c=a$, or $a=b$, according as it is the axis of x, or of y, or of z that is along the optic axis, and the ray-slowness s in the direction (λ, μ, ν) is given by

$$s^2=\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If, then, (p, q, r) be the direction-cosines of the normal to the reflecting surface and the suffixes $(_1)$, $(_2)$ refer to the incident and the reflected extraordinary rays respectively, the principle of least time gives the equations

$$\frac{1}{s_2}\left(\frac{\lambda_2}{a^2}, \frac{\mu_2}{b^2}, \frac{\nu_2}{c^2}\right)+\frac{1}{s_1}\left(\frac{\lambda_1}{a^2}, \frac{\mu_1}{b^2}, \frac{\nu_1}{c^2}\right)=\mathrm{K}(p, q, r), \quad . \quad (3)$$

where K is an undetermined multiplier.

Introducing in place of (p, q, r) the direction-cosines (α, β, γ) of the diameter of the wave-surface that is conjugate to the plane $px+qy+rz=0$, equations (3) become

$$\frac{1}{s_2}(\lambda_2, \mu_2, \nu_2)+\frac{1}{s_1}(\lambda_1, \mu_1, \nu_1)=\mathrm{L}(\alpha, \beta, \gamma), \quad . \quad . \quad . \quad (4)$$

where

$$\mathrm{L}=\mathrm{K}\Big/\sqrt{\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}}.$$

1. As a first result we obtain

$$\begin{vmatrix} \lambda_2, & \lambda_1, & \alpha \\ \mu_2, & \mu_1, & \beta \\ \nu_2, & \nu_1, & \gamma \end{vmatrix}=0,$$

or the diameter of the extraordinary wave-surface, described round the point of incidence, that is conjugate to its section by the reflecting surface, is co-planar with the incident and the reflected extraordinary rays.

2. Let $(\alpha', \beta', \gamma')$ be the direction-cosines of a line in this plane perpendicular to (α, β, γ), then, multiplying equations (4) by α', β', γ' respectively and adding, we obtain

$$\frac{1}{s_2}\sin\chi_2=\frac{1}{s_1}\sin\chi_1,$$

where χ_1, χ_2 are the angles that the direction (α, β, γ) makes with the incident and the reflected rays.

Since the ray-velocities are given by the radii vectores of the wave-surface, this equation expresses that the diameter of the extraordinary wave-surface, described round the point of incidence, that is conjugate to its section by the reflecting surface, is the median of the triangle formed by the incident and the reflected extraordinary rays and a parallel to the reflecting surface.

3. Again multiplying equations (4) by p, q, r respectively and adding, we obtain

$$\frac{1}{s_2}\cos\varphi_2+\frac{1}{s_1}\cos\varphi_1=\mathrm{L}\cos\theta,$$

where φ_1, φ_2, θ are the angles that the normal to the reflecting surface makes with the incident and the reflected rays and the direction (α, β, γ).

Now $\frac{1}{s_1}\cos\varphi_1$, $\frac{1}{s_2}\cos\varphi_2$ are the projections on the normal to the reflecting surface of the ray-velocities along the incident and the reflected rays, and from the above results these are equal. Hence

$$\mathrm{L}=\frac{2}{s_1}\frac{\cos\varphi_1}{\cos\theta}$$

and

$$\frac{\lambda_2}{2\alpha\cos\varphi_1-\lambda_1\cos\theta}=\frac{\mu_2}{2\beta\cos\varphi_1-\mu_1\cos\theta}=\frac{\nu_2}{2\gamma\cos\varphi_1-\nu_1\cos\theta}$$

$$=\frac{\cos\varphi_2}{\cos\varphi_1\cos\theta}=\frac{1}{\sqrt{4\cos^2\varphi_1-4\cos\chi_1\cos\varphi_1\cos\theta+\cos^2\theta}}$$

From the form of these equations it follows that they hold for any system of rectangular co-ordinates.

ABSTRACT.

By the principle of least time it is shown that the diameter of the extraordinary wave-surface described round the point of incidence, that is, conjugate to the reflecting surface, is coplanar with the incident and reflected extraordinary rays and is the median of the triangle formed by these rays and a parallel to the reflecting surface.

The direction-cosines of the reflected ray are then obtained in terms of those of the incident ray and the said diameter of the wave-surface.

XXXIV. *Some Experiments on Tinfoil Contact with Dielectrics.*

By G. E. Bairsto, *M.Sc.*, *B.Eng.*

Received May 7, 1913. Read June 13, 1913.

Although the questions connected with the effect of contact between the conductor and dielectric in the case of condensers have already been to some extent considered, a sufficient number of unsettled points remain which invited further consideration. In the course of a larger investigation on the properties of dielectrics the author was led to direct his attention to these questions, which were touched upon by Dr. J. A. Fleming and Mr. G. B. Dyke* in a recent Paper, and have also been discussed by Mr. Rollo Appleyard.†

The points examined by the author are as follows :—

1. The different effects of pressure and voltage upon tinfoil contact with celluloid as dielectric.
2. The effect of contact upon measurements made with alternating currents.
3. The effect of imperfect contact upon the accumulation of residual charge.

1. *The Different Effects of Pressure and Voltage upon Tinfoil Contact with Celluloid as Dielectric.*

In experimenting with dielectrics it is generally known that if we have tinfoil electrodes pressing against a sheet of dielectrics, and a heavy pressure applied to bring them into more intimate contact, the current a minute or two after the application of the voltage is slightly greater than the current at the time of switching on.

Mr. R. Appleyard has described some experiments on presspahn in which he compares the second minute deflection with the first minute deflection. He finds that for small loads the former is in general greater than the latter, but as the load increases a point is reached at which these deflections become approximately equal. For loads greater than this the second minute deflection was less than that of the first minute. He also found that the apparent resistance gradually decreased with the pressure, and attained a fairly constant value at about 543 grams per square centimetre. He also found that increasing the voltage decreased the apparent resistance.

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* J. A. Fleming, F.R.S., and G. B. Dyke, B.Sc., "On the Power Factor and Conductivity of Dielectrics when tested with Alternating Electric Currents of Telephonic Frequency at Various Temperatures," "Proc." Inst. Eng., Vol. XLIX., p. 351.

† R. Appleyard "On Contact with Dielectrics," "Proc." Phys. Soc., 1905, Vol. XIX., p. 724-737; "Phil. Mag." Vol. X., p. 485; and "Science Abstracts," Vol. VIII.A, No. 2099, 1905.

The following experiments, which go into the matter in more detail, were made with the object of trying to elucidate the cause of this increase of current after the first switching on of the current, by studying the effect at very low pressures as well as at the point at which perfect contact is obtained, and by allowing sufficient time (in most cases hours) for the current to reach its final steady value. In none of Mr. Appleyard's experiments does the increase of current amount to more than about 6 per cent., whereas by choosing a suitable dielectric, the author has been able to obtain increases of more than 40 per cent. The reason for this lies in the fact that with presspahn, the material used by Mr. Appleyard, the decrease in current due to absorption is of a greater order of magnitude than the conduction current, whereas celluloid, the dielectric which was used for the following experiments, has a much greater conduction current than displacement current, and so the effect of absorption has nothing like the influence it has with presspahn. Celluloid is therefore a much more convenient dielectric for use in investigating the effects under notice.

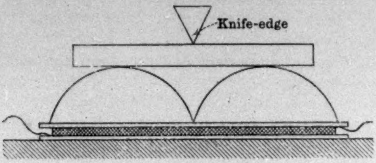

FIG. 1.

A single sheet of celluloid having an area of about 300 sq. cm. and thickness of 0·0762 cm., was placed between two well-smoothed tinfoil electrodes having an area of 205 sq. cm. each. It was placed in a testing machine with a thick piece of millboard underneath and one above it, and the whole insulated with several layers of well-dried manilla paper. Two half-round steel pieces (*see* Fig. 1) with planed undersides served to distribute the pressure applied by the knife-edge of the testing machine to another flat piece of iron placed on top of both, the knife-edge being, of course, in a symmetrical position.

The E.M.F. used was 100 volts, and the current was measured by the direct deflection method.

The temperature during the whole course of experiments remained between the limits of 15·7° and 16·3°C.

Table I. gives a set of results for very low, and for moderate pressures, each test being given sufficient time to allow the current to come to an absolutely steady value, after the first instantaneous increase had taken place. It will be seen that at low pressures the time required is short, *i.e.*, of the order of 5 to 15 minutes, but that as the pressure is gradually increased the time now required to reach a steady maximum is of the order, not of minutes, but of hours. Moreover, the increase

Table I.

Weight in lbs.	Time.	10^{-8} amperes per sq. cm.	Weight in lbs.	Time.	10^{-8} amperes per sq. cm.
Approx. 5	...	1·16	250	½ min.	10·82
15	½ min.	2·38		1 ,,	10·85
	1½ ,,	2·43		2 ,,	11·04
	2 ,,	2·46		5 ,,	11·24
	5 ,,	2·47		10 ,,	11·26
25	½ ,,	3·19		15 ,,	11·3
	1 ,,	3·21		20 ,,	11·31
	5 ,,	3·22	300	½ ,,	11·38
35	½ ,,	3·63		1 ,,	11·41
	5 ,,	3·64		2 ,,	11·58
45	½ ,,	4·07		5 ,,	11·59
	1 ,,	4·12		15 ,,	11·77
	2 ,,	4·15		30 ,,	12·04
	5 ,,	4·16		45 ,,	12·47
50	½ ,,	4·39		1 hour	12·70
	2 ,,	4·48		1¼ ,,	12·81
	5 ,,	4·50		1½ ,,	12·92
60	½ ,,	4·70		2 ,,	13·28
	5 ,,	4·82	400	½ min.	13·97
	10 ,,	4·84		5 ,,	14·22
	20 ,,	4·87		15 ,,	14·62
70	½ ,,	5·24		½ hour	15·37
	2 ,,	5·27		1 ,,	16·38
	5 ,,	5·35		1¼ ,,	16·92
	10 ,,	5·36		2 ,,	17·70
80	½ ,,	5·56	500	½ min.	18·75
	2 ,,	5·59		1 ,,	18·91
	10 ,,	5·67		5 ,,	19·14
150	10 ,,	8·1		15 ,,	19·5
200	10 ,,	9·82		30 ,,	19·87
225	½ ,,	10·18		1½ ,,	20·8
	1 ,,	10·22		2 ,,	21·4
	5 ,,	10·48		3 ,,	22·15
	10 ,,	10·53		3¾ ,,	22·4
	15 ,,	10·53		24 ,,	22·3

in current, at first small, becomes larger and larger as the pressure rises.

If we plot the current to pressure, as has been done in the inset to Fig. 2, we find that from 25 lb. upwards it follows a linear law :—

$$\text{Current} = a + b\ (\text{pressure}).$$

The short curved portion at the beginning of the straight line would seem to indicate that about 25 lb. was required to bring the tinfoil into a settled down condition on to the dielectric.

When a pressure of 300 lb. had been reached a break was made in the experiments. The pressure was still kept on, but the voltage switched off and then re-applied after the lapse of 24 hours. Table II. gives the results of this test. It will be seen

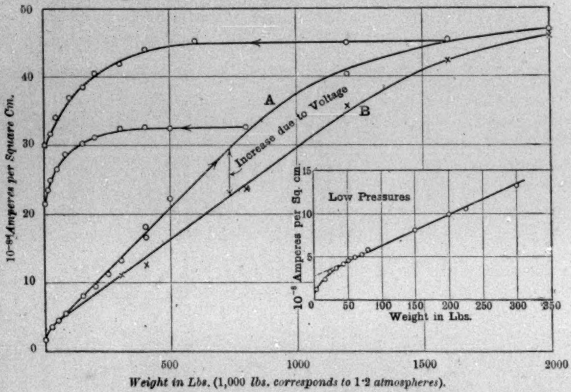

FIG. 2.—VARIATION OF CURRENT WITH PRESSURE.

TABLE II.—*Repeat Test.*

Weight in lbs.	Time.	10^{-8} amperes per sq. cm.
300	½ min.	11·26
	2 ,,	11·49
	5 ,,	11·58
	20 ,,	11·97
	30 ,,	12·22
	45 ,,	12·72
	1 hour	12·92
	1¼ ,,	13·14
	1½ ,,	13·48

that the current starts at practically the same value it initially had when the pressure of 300 lb. was first applied, and also gradually increases up to its previous maximum value. Moreover, the time required is practically the same in both cases, viz., about two hours. The current does not, of course, commence at exactly the same value, but at a point slightly lower, because in Table I. the dielectric had already reached a steady condition under voltage at a lower pressure, before the 300 lb. pressure was applied.

The above tests clearly show that voltage has quite as much effect as pressure in bringing about an intimate contact with tinfoil electrodes, and would also seem to indicate that each acts independently of the other.

TABLE III.

Weight in lbs.	Time.	10^{-8} amperes per sq. cm.	Weight in lbs.	Time.	10^{-8} amperes per sq. cm.
400	½ min.	12·63	1,200	1 min.	35·6
	5 ,,	13·20		5 ,,	35·8
	15 ,,	13·65		10 ,,	36·0
	30 ,,	14·20		15 ,,	36·2
	1 hour	14·49		30 ,,	36·6
	1½ ,,	15·5		1 hour	38·1
	2 ,,	16·1		3 ,,	40·1
	2½ ,,	16·2		5 ,,	40·4
	3½ ,,	16·7	1,600	½ min.	42·3
800	½ min.	22·15		1 ,,	42·4
	1 ,,	22·35		5 ,,	42·7
	10 ,,	23·1		15 ,,	42·8
	20 ,,	23·6		30 ,,	43·4
	30 ,,	23·95		1 hour	44·5
	45 ,,	24·6		1½ ,,	45·1
	1 hour	25·25		2½ ,,	45·3
	1¼ ,,	25·8		3½ ,,	45·4
	2¼ ,,	28·1		24 ,,	45·45
	2½ ,,	28·75	2,000	½ min.	45·8
	3 ,,	29·85		10 ,,	46·0
	4 ,,	31·9		30 ,,	46·4
	4½ ,,	32·1		1 hour	46·8
	5 ,,	32·6		1½ ,,	46·9
				2½ ,,	46·9

To test this still further, both the pressure and voltage were removed, and the whole system left to itself for a day, and another set of experiments, embodied in Table III., made, in which the pressure was applied in a series of equal increments of 400 lb., starting at 400 lb., and going up to 2,000 lb.

Fig. 3 shows how the current increases with the time for these different increments of pressure. In continuation of the low pressure series, it will be seen that the time required for the current to reach its final value is longer and longer, reaches a maximum value of five hours at 800 lb., and afterwards decreases as the pressure rises to the point at which intimate contact is made with the dielectric.

In Fig. 2, curve A, the final values of the current are plotted against pressure; at 2,000 lb., the curve has practically become flat.

At the end of the 800 lb. test the system was again left to itself for 24 hours without pressure or voltage being on, and a repeat test made at 800 lb. The results are denoted by crosses on Fig. 3. As with the 300 lb. repeat test, they lie on the same curve, and moreover the current, after the instantaneous increase, is practically the same as in the first case, showing that the pressure increments are so large that we may assume that for any one of the five pressures the current at the end of the instantaneous increase is the same as if the pressure had been applied immediately without any intermediate step.

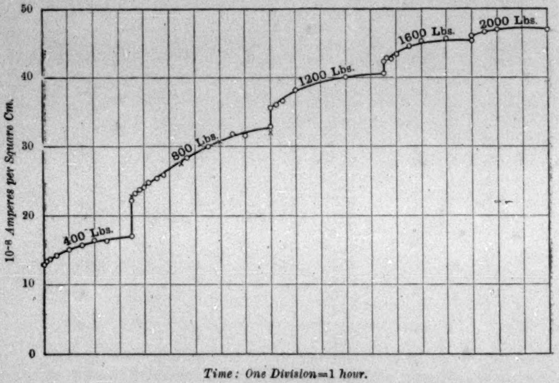

FIG. 3.

Using this information, it is instructive to tabulate what may be called the instantaneous, and the secondary increments in current, and their percentages of the total increments. This has been done in Table IV. Fig. 4 shows how the ratio of instantaneous increase to total increase and of secondary increase to total increase vary with the successive pressure increment. The secondary increase, at first a small percentage of the total increase, becomes larger and larger, until at high pressures it accounts for practically all of the increase.

TABLE IV.

Weight in lbs.	Current in 10^{-8} amperes/sq. cm.			Increase of current.			Instantaneous increase. / Total increase.	Secondary increase. / Total increase.
	Initial.	After the instantaneous increase.	Final.	Instantaneous.	Secondary.	Total		
400	2·73	12·63	16·7	9·90	4·07	13·97	0·71	0·29
800	16·7	23·65	32·2	6·95	8·55	15·5	0·45	0·55
1,200	32·2	35·6	40·4	3·4	4·8	8·2	0·41	0·59
1,600	40·4	42·3	45·45	1·9	3·15	5·05	0·38	0·62
2,000	45·45	45·80	46·90	0·35	1·10	1·45	0·24	0·76

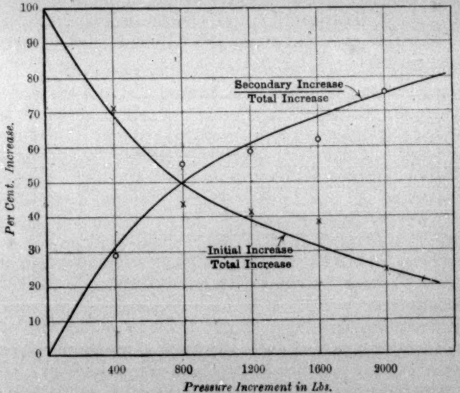

FIG. 4.—SHOWING THE PERCENTAGE INCREASES OF CURRENT FOR SUCCESSIVE PRESSURE INCREMENTS.

Considering now the secondary increase as a function of the pressure on the tinfoil, Fig. 5 shows how these two quantities are connected. At low pressures exceedingly small, the secondary increase becomes larger and larger as the pressure increases, attains a maximum value at about 800 lb., and gradually falls to a small value again. In Table V., column 2 corresponds to that of column 3 of Table IV., and in column 4 are given the values of the ratio of the secondary increase to the current after the instantaneous increase took place, or, in other words, the ratio of the difference between the final and initial deflections of the galvanometer to the initial deflection. The variation of this quantity with pressure is also given in Fig. 5. It will be seen that at about 650 lb. it attains a value of over 40 per cent.

TABLE V.

Weight in lbs.	Current in 10^{-8} amperes per sq. cm.		Per cent. increase $\frac{(2)-(1)}{(1)}\cdot 100$.
	Before secondary increase (1).	After secondary increase (2).	
300	11·26	13·48	20·0
400	12·63	16·70	32·0
800	23·65	32·20	36·0
1,200	35·60	40·4	13·5
1,600	42·30	45·45	7·5
2,000	45·80	46·90	2·5

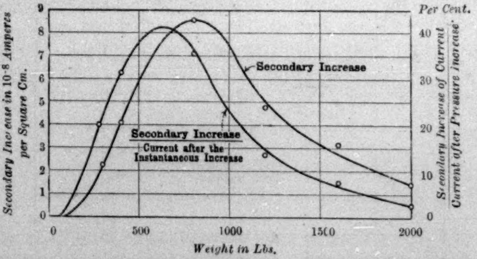

FIG. 5.—VARIATION OF SECONDARY INCREASE OF CURRENT WITH PRESSURE.

It is evident from the above results that the secondary increase is due wholly and solely to the application of the voltage, and we may construct as in Fig. 2 a curve B, such that for any pressure its ordinate is equal to the current at the first minute of application of the voltage, and the difference between it and curve A gives the secondary increase due to the voltage.

An ordinate of curve B measures the degree of intimacy of contact due to the pressure, and the difference between A and B the further increase due to the action of the voltage.

At two stages in the last-mentioned series of experiments, viz., at 800 lb. and 1,600 lb. respectively, the pressure was gradually reduced down to zero again. The current-pressure curves are given in the two upper curves of Fig. 2. It is to be noted in each case that the current does not begin to fall away until the pressure has been reduced to about one-third of its value, and that also in both cases the current when the pressure has been reduced to zero, was about 65 per cent. of its value at full pressure.

For convenience the forces applied to the tinfoil have been expressed in pounds weight applied by the knife-edge of the testing machine. Since the area over which the pressure was distributed amounted to 290 sq. cm., 1 lb. corresponds to a pressure of 1·26 grams per square centimetre, so the pressure exerted by 2,000 lb. on the tinfoil is equal to 2,500 grams per square centimetre, or since 1 kilogram per square centimetre is equal to 0·97 atmosphere, this corresponds to a pressure of 2·4 atmospheres. This is the pressure that was required to bring the tinfoil into an intimate contact with the celluloid.

The effects described above may be explained as follows :—

Let (*a*) Fig. 6 represent a section, very much magnified, taken through the dielectric and its tinfoil electrodes. It represents a number of hollows with corresponding humps full of air, current being able to pass through the bases of the hollows. The effect of pressure will be twofold. Firstly, to make one hump into two, say, as in Fig. (*b*), or secondly to flatten it as in (*c*). In the first case we should have an instantaneous increase of current, as soon as the pressure is applied. In the second case, there will be little increase of area of electrode in contact with the dielectric, and therefore very little instantaneous increase of current. If, however, we consider the distribution of the stream lines of current we shall have a system as shown in the figure, such that at a point A, not far from the base of the hump, there will be a P.D. between that point of the dielectric, and B. a point on the dielectric vertically above it. This P.D. acting across the very thin film of air will give rise to a large potential gradient, and therefore a large local mechanical force. This force gradually pulls more and more of the surface of the tinfoil down on to the dielectric, at first quickly, because the angle A P B is acute, and then slowly, for A.P.B. becomes as in Fig. (*d*), obtuse. The current therefore increases quickly at first, and then reaches its maximum value more slowly, which is just what we have seen by experiment (Fig. 3) actually happens.

The contact is therefore made up of these two parts, the primary increase, and the secondary increase due to the action of the voltage. The extent to which each is important will depend on the magnitude of the applied pressure. Now it will be obvious from geometrical considerations that the angle at the base of hump in (*b*) will be smaller than in (*a*). So at low pressures the secondary increase will be small, and will increase as the pressure increases, and the hump becomes smaller. In a series of tests, therefore, made with successive equal increments of pressure, the ratio of the secondary increase of current to the total increase of current should increase with the pressure increment. This agrees with the results depicted in Fig. 4.

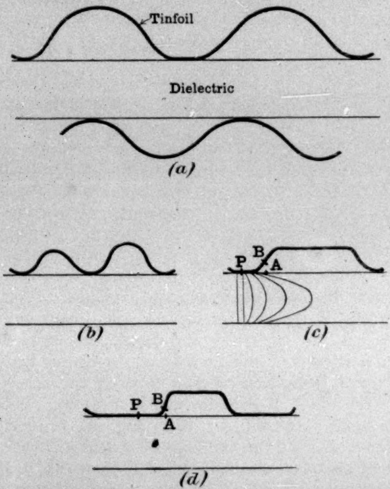

FIG. 6.

Again, considering the absolute value of the secondary increase, and its variation with applied pressure, we have seen that it is small at small pressures, and increases as the pressure rises. On the other hand, when the pressure is sufficiently large enough to bring the tinfoil into perfect contact with the dielectric, there can be no secondary increase, because all the tinfoil is now in contact. This we have seen in the above is what takes place (*see* Fig. 5). Between these two pints of low and of high pressures there is a maximum which will depend upon a number of factors, the principal of which will probably be the nature, size and shape of the humps, and the mechanical properties and thickness of the tinfoil. Absorption will also, of course, play a part in reducing the value of the secondary increase as the pressure rises.

We have seen that a repeat test made at the same pressure after the voltage has been removed for some time, leads to practically the same current-time curve. This is to be explained by the natural resilience of the tinfoil causing it to spring back again. This effect would only be wiped out by applying a very large pressure, and then a repeat test made at the original pressure would lead to quite different results.

Since the secondary increase is due to the presence of the voltage, it follows that when the system has become steady at a given pressure below that at which intimate contact takes place, and we now decrease the voltage we should expect to find an increase in the apparent resistance. This is illustrated by Table VI., and agrees with Mr. Appleyard's observations previously mentioned.

TABLE VI.—*Variation of Apparent Specific-resistance of the Dielectric with Voltage after the Pressure had been brought up to* 1,600 *lb.*

Volts.	10^{-8} amperes per sq. cm.	Ohms per cu. cm.
97	45·5	$0{\cdot}0283 \times 10^{12}$
74	34·2	0·0287 ,,
49	21·5	0·0298 ,,
24	9·98	0·0316 ,,

2. *The Effect of Contact upon Measurements made with Alternating Currents.*

We have seen that in direct-current measurements considerable errors are liable to be made in deducing the specific resistance from the current flowing through the dielectric when the contact is a tinfoil one, but when we are dealing with the losses in dielectrics for alternate currents of high frequency it would appear that the error due to bad contact is very much diminished. Dr. J. A. Fleming and Mr. Dyke* have shown theoretically that it should be very much diminished in these circumstances. They show that by considering the condenser with its tinfoil armatures as equivalent to a thin air condenser in series with a high resistance R equal to the resistance *per se* of the condenser, and deducing an expression for the effective conductance of the combination for alternating currents of a given frequency, the presence of the thin air film has very little influence on the observed, or apparent conductance of the system for alternating currents of that frequency.

We shall here consider a more general case, and suppose the system equivalent to a condenser of dielectric having capacity C, and conductance S, in series with a thin air condenser *c* (Fig. 7).

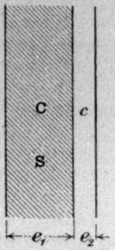

FIG. 7.

Let i=current flowing through the dielectric, e_1 and e_2 =E.M.F.s across C and c respectively, and E sin pt = the applied E.M.F.

Then
$$i = C\frac{de_1}{dt} + Se_1$$
$$= c\frac{de_2}{dt};$$
$$\therefore \quad i(C+c) = Cc\frac{de}{dt} + Sce_1$$
$$(C+c)\frac{di}{dt} = Cc\frac{d_2e}{dt^2} + Sc\frac{de}{dt} + Sc\frac{de_2}{dt},$$

---

* "Journal" of the Institution of Electrical Engineers, 1912, Vol. XLIX., p. 351.

or $$(C+c)\frac{di}{dt}+Si=-CcEp^2\sin pt+Sc\cos ptEp$$
$$=-Ecp\sqrt{C^2p^2+S^2}\sin(pt-\phi),$$

where $$\tan\phi=\frac{S}{Cp},$$
$$\therefore\quad i=\frac{Ecp\sqrt{C^2p^2+S^2}}{\sqrt{S^2+(C+c)^2p^2}}\sin(pt-\phi-\theta). \qquad (1)$$

where $$\tan\theta=\frac{(C+c)p}{S}.$$

The power therefore wasted in the system is—

$$W=-E^2cp\sqrt{\frac{S^2+C^2p^2}{S^2+(C+c)^2p^2}}\cdot\frac{\cos(\phi+\theta)}{2}$$
$$=\frac{c^2p^2}{S^2+(C+c)^2p^2}\cdot\frac{SE^2}{2}.\quad\ldots\ldots\ldots\qquad (2)$$

The equivalent conductance is therefore

$$=\frac{c^2p^2}{S^2+(C+c)^2p^2}S.\quad\ldots\ldots\ldots\qquad (3)$$

Also we have the total admittance from (1)—

$$Y=\frac{cp\sqrt{C^2p^2+S^2}}{\sqrt{S^2+(C+c)^2p^2}},$$

and since $Y^2=$ (equivalent conductance)$^2+$(apparent capacity)^2 p^2, we get on reduction the following expression for the apparent capacity—

$$\frac{c[S^2+Cp(C+c)p]}{S^2+(C+c)^2p^2}.$$

Since the value of S/Cp for most dielectrics at ordinary temperature is of the order of 0·01, or less, and S/cp will therefore be still smaller, we deduce the following approximate expressions :

$$\left.\begin{aligned}\text{Equivalent conductance} &=\left(\frac{c}{C+c}\right)^2 S\\ \text{Equivalent capacity} &=\left(\frac{c}{C+c}\right)C\\ \text{Apparent power factor} &=\left(\frac{c}{C+c}\right)\frac{S}{Cp}\end{aligned}\right\}\quad\ldots\qquad (4)$$

In Table VII. are given the results of two tests made in order to see what is the actual difference that can be obtained with a tinfoil contact of this kind. They refer to a certain condenser made up of celluloid in very thin sheets 0·0146 cm. in thickness.

TABLE VII.
(1 Bimho $=10^{-12}$ mho.)

Test.	1. Pressure on.			2. Pressure off.		
Frequency.	Capacity C_P in micro-micro-farads.	$\frac{S_P}{C_P p}$.	Conductivity σ_P in Bimhos per cm. cu.	Capacity C_0 in micro-micro-farads.	$\frac{S_0}{C_0 p}$.	Conductivity σ^0 in Bimhos per cm. cu
920	1,780	0·0228	59·9	1,670	0·0211	52
2,760	1,755	0·0192	149·0	1,665	0·0182	134
4,600	1,730	0·0183	233·0	1,630	0·0174	211

In test No. 1 the condenser was squeezed together with such a pressure as by the previous experiments, would give an approximately intimate contact, whereas in test No. 2 it was under no pressure, but was lightly bound round with tape so as to just hold it together. The measurements were made with the Fleming and Dyke variable capacity bridge. The

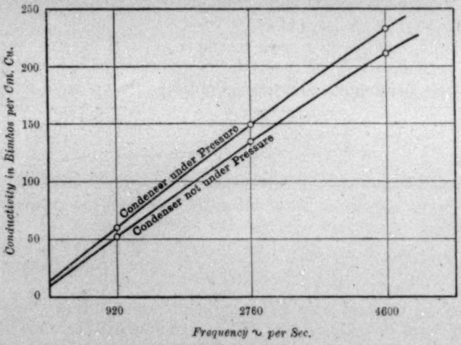

FIG. 8.—SHOWING THE EFFECT OF CONTACT UPON MEASUREMENTS MADE WITH ALTERNATE CURRENTS IN CELLULOID CONDENSER.

results are also depicted in Fig. 8, and expressed as percentages in Table VIII. The suffixes P and 0, refer to pressure on, and pressure off, respectively.

TABLE VIII.

Frequency.	Per cent. variation of		
	$\frac{C_P}{C_0}$.	$\frac{S_P}{C_P p} \Big/ \frac{S_0}{C_0 p}$.	$\frac{\sigma_P}{\sigma_0}$.
920	6·5	6·0	15·0
2,760	5·5	5·5	10·5
4,600	5·0	5·0	10·0

Now, the influence of the bad contact is two-fold ; firstly, it decreases the apparent capacity by inserting in series with the condenser under test a very large, but still finite, air condenser. This by the above equations causes a decrease in the apparent conductance. Secondly, it decreases the magnitude of that component of the conductivity which is independent of the frequency, *i.e.*, the purely ohmic conductivity, because of the decrease in the area of contact.

The equations (4) give us a means of separating the two effects, for according to them the percentage change in the equivalent conductance is twice that of the equivalent capacity. Now the average percentage change of C is about 5·5, and this is just about half the change in the conductivity at 2,760 and 4,600 ∿. At 920 ∿, however, there is a difference at about 5 per cent. This is the amount of error due to the second effect, *i.e.*, bad contact, decreasing the ohmic part of the conductivity. At lower frequencies still this would be still larger. Hence we can say that for telephonic frequencies the main influence of a bad contact is to lower the observed conductivity by inserting a capacity in series with the condenser under test.

It must be understood that the above case is a very extreme one, since the dielectric was very thin, and the contact made very much worse than would occur in practice. A further test, in which the tinfoil after being carefully smoothed and rolled on to the celluloid, and the whole tightly bound with tape between glass plates, and then wedges inserted between the tape and the glass, gave a maximum change of C of 2·5 per cent. and a change in σ of 4·5 per cent. when tested in this condition and then under pressure. For dielectrics of greater thickness the change would be less.

3. *The Effect of Imperfect Contact upon the Accumulation of Residual Charge.*

In a well-known lecture experiment, to show the existence of residual charges in dielectrics, a Leyden jar is charged to a high voltage for a short time, and then by means of a pair of discharging tongs is discharged again. On being left to itself for a time and properly insulated, a further discharge almost as strong, as indicated by the spark, may easily be obtained.

In view of the previously described experiments, the question presents itself as to whether this second discharge is entirely due to the presence of a residual charge in the glass, or whether we are not really dealing with a case of very bad contact between the tinfoil armatures of the jar and the glass dielectric.

TABLE IX.

Time.	Voltage of residual charge.				
	Both electrodes tinfoil. (1)	Both electrodes tinfoil. Better contact. (2)	One electrode tinfoil. Second electrode mercury. (3)	Both electrodes uncleaned mercury. (4)	Both electrodes clean mercury. (5)
5 secs	50·5	41·5	29·0	22·0	18·0
10 ,,	53·0	...	36·0	28·5	21·0
15 ,,	...	44·5	...	...	22·0
20 ,,	53·5	...	40·0	31·5	...
30 ,,	49·0	47·0	42·0	32·5	22·0
1 min.	41·0	43·5	41·5	33·0	18·0
1½ ,,	33·5	39·5	39·5	31·0	15·5
2 ,,	27·5	34·5	38·0	29·0	13·5
3 ,,	22·0	27·5	36·0	25·5	9·5
4 ,,	18·0	23·0	32·0	22·0	7·0
5 ,,	14·0	19·0	29·0	19·0	4·5
6 ,,	...	15·5	...	16·0	...
7½ ,,	7·0	12·0	21·0	13·5	2·5
10 ,,	4·0	8·0	16·0	10·5	...

Table IX gives the results of some experiments made to decide this point. A glass tube about 1 in. in diameter and 4 in. long was used as the condenser. It was carefully insulated on a paraffin block, and could either be covered with tinfoil or else filled with mercury and placed inside another and larger tube full of mercury so as to obtain two electrodes in intimate contact with the glass. This tube was connected up as shown (Fig. 9) with a two-way mercury switch made out of a paraffin block, arranged so as to charge the glass condenser with a small insulated battery of secondary cells of about 100 volts for a given time, and then on throwing over the switch, to momentarily short-circuit it, and finally by removing the cross-connecting wires to leave the tube connected with an electrostatic voltmeter which indicated the growth of the residual charge. The voltmeter was one of Ayrton and Mather's, and had a very short period. It gave about 26 cm. deflection at 2 metres for 100 volts. Its insulation as well as that of the whole circuit was considerably better than that of the glass tube itself.

In the first column of Table IX are given the values of the P.D. between the tinfoil armatures at different times after a momentary short circuit, the tube having been previously charged for 10 minutes at a voltage of 99. The results have been plotted in curve 1 of Fig. 10. The residual charge grows very rapidly, and in about a quarter of a minute has reached its maximum potential. This maximum potential of the recovered charge is more than half that of the original potential. It then dies away as the charge leaks through the dielectric. In this case the tinfoil both inside and out were simply pressed into contact with the glass tube. In the case of test No. 2 the contact was made a better one by binding the tinfoil with fine wire on the outside, and thoroughly pressing down the inside armature, and here we see that the residual charge does not attain quite as high a value, only 47 volts. In the next test, No. 3, the outside tinfoil was left in position, but the inside coating removed and replaced with clean mercury. The maximum potential of the charge recovered is still lower—42 volts.

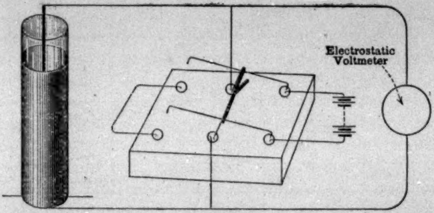

FIG. 9.

In case No. 4 both electrodes were formed of mercury that had been standing about the laboratory for some months, and was very dirty and dusty. In this case we have a still further drop of maximum voltage to 33.

Finally, in the fifth test, both electrodes were made of mercury that had been thoroughly cleaned, and we see that the maximum voltage of the recovered charge, in this case the true residual charge, is now only 22 volts, *i.e.*, is just half what it was with tinfoil armatures.

The source of these differences, as we have said, lies in the degree of contact produced between the armatures and the dielectric. If part of the tinfoil is not in contact with the glass, the portion of the condenser corresponding to it is not discharged at the moment of short circuit, but the charge in it gradually comes out afterwards, and leaking along the surface of the glass, charges up the tinfoil and thus gives rise to an increased charge over and above the residual charge left in the dielectric.

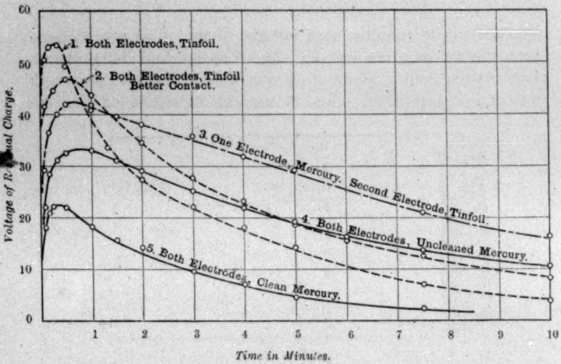

FIG. 10.—SHOWING THE EFFECT OF CONTACT ON THE ACCUMULATION OF RESIDUAL CHARGE.

Dielectric: Glass tube charging volts, 99. Time of charge, 10 minutes. Momentary Short Circuit.

There is one point in connection with Fig. 10 that requires further explanation. It will be noticed that as we proceed from curve 5 to curve 3, *i.e.*, as the contact becomes worse, the rate of fall of potential due to leakage through the glass becomes smaller and smaller. This necessarily follows from the law—

$$\frac{dV}{dt} = -\frac{V}{CR},$$

where V is the potential at any time t, and C and R, the capacity, and insulation resistance of the condenser. C is approximately constant, and R will be inversely proportional to the area of contact,

$$\therefore \quad \frac{dV}{dt} \propto V(\text{area of contact}),$$

that is to say, the rate of fall of the logarithm of the potential is proportional to the area of contact. But when we make the contact still worse by employing a loose tinfoil contact, C will be very materially reduced, on account of the thin film of air in the humps of tinfoil. Moreover, the dielectric constant of this air condenser is only about one-eighth of that of the glass.

The net result of this will be an increase in the rate of fall of potential more than the decrease due to lack of contact, and hence the curves 1 and 2 eventually fall below those of 3, 4 and 5.

In conclusion, we may say that if we have a condenser with tinfoil armatures, as in the Franklin jar with its rigid metallic coatings, the recovery of a residual charge will be obscured by the presence of creeping surface charges coming out of the undischarged portions of the dielectric.

I take this opportunity of expressing my thanks to Dr. J. A. Fleming, F.R.S., for his advice and suggestions in this work, and to Prof. Cormack for placing the testing machine at my disposal. The experimental work has been carried out in the Research Laboratory of the Electrical Engineering Department of University College, London.

ABSTRACT.

This Paper describes some experiments showing how the accuracy of the different kinds of electrical measurements that are made on condensers is influenced by the use of an imperfect tinfoil contact.

1. In connection with the measurement of the direct-current conductivity of a condenser having tinfoil armatures, the experiments of Mr. Appleyard ("Proc. Phys. Soc.," 1905, Vol. XIX., p. 724), in which the current a minute or two after the first switching on of the current was greater than that at the time of switching on, are referred to. These experiments go into the matter in more detail. By choosing a suitable dielectric—celluloid, which has a

conduction current of a greater value than the rate of change of displacement current—it was found possible to greatly increase the magnitude of these secondary increases in current.

At very low pressures there is very little increase of current, and moreover what little increase there is is over in a few minutes, but as the pressure is increased the secondary increase gradually becomes larger and takes longer to attain its maximum value. The maximum effect was reached at about 750 grammes/sq. cm., when the increase of current amounted to 40 per cent., and the time required 5½ hours. At very large pressures, when the contact becomes an intimate one, the increase of current becomes smaller again and the time required also smaller.

If the pressure is left on and the voltage removed for some hours a repeat test follows the same course, the current starting at the same value and attaining the same maximum value. Voltage has, therefore, quite as much effect as pressure in bringing about an intimate contact and acts independently of it.

By considering the geometry of the tinfoil humps, an explanation of these various effects is given, and the different ways in which the pressure and the voltage increase the degree of contact between the dielectric and the tinfoil armatures are described.

2. While considerable errors are liable to be made in deducing the specific direct current conductivity of a dielectric between tinfoil armatures, the same is not true for measurements of the alternating-current conductivity. The influence of the bad contact is twofold. Firstly, it decreases the apparent capacity by inserting in series with the condenser under test a very large but still finite air condenser. This causes a decrease in the measured conductance. Secondly, because of the decrease in area of contact, it decreases the magnitude of that component of the conductivity which is independent of the frequency—*i.e.*, the purely ohmic conductivity.

By considering the system as equivalent to a leaky condenser in series with a very large capacity due to the air film, expressions are deduced for the equivalent capacity, conductance and power factor, and these expressions furnish us with the means of separating out the two above effects.

It is shown experimentally, even under the worst possible circumstances, the dielectric being only lightly bound up with the interleaved tinfoil, that for telephonic frequencies the maximum difference between the observed conductivity and true conductivity is 15 per cent. and of capacity is 5 per cent. With the condenser tightly bound with tape and wedges of wood inserted, the maximum difference was only 4·5 per cent. in the conductivity and 2·5 per cent. in the capacity.

3. Finally, the influence of imperfect contact upon the accumulation of residual charge is considered. It is shown that if we have a condenser with tinfoil armatures, as, for instance, in the Franklin jar, with its rigid metallic coatings, the recovery of a residual charge is obscured by the presence of creeping surface charges coming out of the undischarged portions of the dielectric leading to an apparent residual charge much more than the true residual charge left in the dielectric.

DISCUSSION.

Dr. J. A. **Fleming** thought the Paper contained much valuable information. He emphasised the difficulty and importance of getting rid of the air film. For many dielectrics, such as glass or sulphur, a considerable pressure could not be applied. In this case the tinfoil could be squee-geed on to the dielectric when the condenser was made or the condenser could be put in a vacuum subsequently. For a constant condenser it was also necessary that there should be no chemical action between the metal plates and the dielectric such as occurred, for instance, in the case of copper foil and celluloid. He thought Mr. Bairsto's experiments on residual charges were very interesting. Theorists had attributed the whole effect to the properties of the dielectric and not to the bad contact between it and the electrodes. The Paper was also useful in pointing out the pitfalls of experimental work on the subject.

Mr. R. **Appleyard**: It is very gratifying to find that Mr. Bairsto has been able to confirm the results which I had the privilege of describing to the Physical Society in the Paper of June, 1905, which he has cited. Judging from some remarks of the author in his present communication, I think he must have overlooked a Paper on dielectrics, read before the Physical Society in May, 1894, in which I set forth the results of tests on celluloid sheets, similar to those now adopted by Mr. Bairsto. In that Paper the effect, upon dielectric resistance, of increasing the testing voltage is clearly shown for the case of hard metallic electrodes, and the effect of residual charge in celluloid sheets is also considered. Mr. Bairsto has attacked an intricate and fascinating group of phenomena in an ingenious and helpful manner, and he has given us a Paper of considerable value. I am glad that he has adhered to the direct-reading method of examining the changes in dielectric resistance. He probably has some good reason for maintaining the current on until the steady reading has been approached, but I would recommend him to examine the merits of taking the reading after a prescribed period—say, of one minute—after the first switching on of the current, and to work out the megohms from that reading as a standard result. Resistances corresponding to subsequent minutes can of course be worked out also if desired. The truth is that the steady state is reached only after infinite time; and, moreover, a number of noteworthy things are happening during the first minute. In the second part of his Paper Mr. Bairsto embarks upon an investigation of the phenomena of alternating currents which has no well-defined relation to the results observed by him in the first part of his research. The conductance to direct currents referred to in the first part must be distinguished from the conductance leakance to alternating currents investigated in the second part. Conductance leakance is simply a coefficient used in representing the watts lost in the dielectric with alternating currents, as shown in his equation (2). In the direct-current case, with tinfoil, he is measuring the dielectric resistance, plus the bad contact resistance in series with it. In the alternating-current case he is measuring a dielectric-hysteresis effect, which might conceivably be measured even though there were complete discontinuity between the dielectric and its electrodes; or, in other words, even though the surface-contact were of infinite resistance. Again, can the author be sure that the increase of capacity shown in Table VII. is not due to diminution of distance between the electrodes, resulting from the increase in mechanical pressure. I wish he would extend the research to obtain positive measurement of the increase of pressure mentioned on p. 309. The diminution of resistance with increase of voltage is not restricted to tinfoil. It is shown in my Paper of October, 1894, to be strongly marked in the case of hard brass plates. The result would, therefore, appear to be due rather to change of contact-resistance than to change of capacity, for the change of capacity in this case is minute. The author's experiments on residual change are of very great value, and they suggest a most useful line of research.

Mr. E. H. **Rayner** remarked that it was always assumed that pressure had no direct influence on the properties of the dielectric, but simply improved the contact between the electrode and the dielectric. Pressure might directly decrease the resistance of the dielectric in the same way as a wet sponge would have its resistance diminished by pressure.

Dr. A. **Russell** agreed with Mr. Rayner's remarks. Celluloid was far from being homogeneous. Mr. Appleyard, he remarked, had shown that the insulation resistance of a condenser is not constant, but a function of the applied voltage, decreasing as this is increased. Hence, Ohm's law cannot be applied. He pointed out that the author had represented his condenser as a capacity shunted by a resistance, all in series with another capacity (that of the air film). This would make the leakage current zero on the direct current test. The representation should be two condensers in series all shunted by the leakage resistance.

Mr. G. L. **Addenbrooke** remarked that he had tried in a rougher way some of the same experiments now described. He remarked that Mr. A. Campbell had suggested blackleading the surfaces of the dielectric to do away with the effect of the air film. He also emphasised the importance of heating in the dielectric due to energy losses. For celluloid which had a temperature coefficient of 10 per cent. per degree this was an important consideration.

Mr. W. **Duddell** remarked that the Paper showed the importance of a thin air film in the determination of the capacity and conductivity of a condenser. For standard condensers the maker knew that if the air was not excluded between the tinfoil and the dielectric the capacity would not remain constant. If a high voltage were applied to such a condenser and left on for some time, the capacity afterwards would be found to be permanently altered. Even with as low a voltage as 200 volts brush discharges could take place from the tinfoil into the air film. This brush discharge into the air film at 200 volts could be actually seen in a dark room if one electrode' be replaced by water, while the other one was tinfoil and the dielectric was mica. This brush discharge was also an important consideration in the slots for the windings of dynamos and alternators.

Prof. C. H. **Lees** expressed his interest in the third section of the Paper on the residual discharge. He would like the author to see whether tinfoil electrodes under pressure gave the same residual discharge curve as the mercury electrodes.

Mr. W. **Eccles**: Mr. Bairsto's excellently planned experiments go far towards removing the obscurity surrounding the subject. The explanation of the effects of pressure and voltage on the contact between electrode and dielectric will probably be widely accepted; but it seems to me that this purely mechanical explanation, based on the distortion of air-filled blisters, is somewhat insufficient. The experiments show that within limits the recovery from applied pressure and voltage is so complete as to suggest the perfect elasticity of the combination, yet, somewhat in contradiction to this, there is a very slow development or decay of the secondary or electro-mechanical effect. The author's mechanical explanation does not appear to account for this viscous part of the phenomenon. I suggest that it may be largely accounted for by supposing that after a voltage is applied there occurs a creeping of charge over the dielectric surface from the tinfoil contact areas (just as described by the author in discussing residual charge), and, therefore, a gradual and not an instantaneous increase in the area of tinfoil attracted into contact. This supposition added to the author's mechanical explanation appears to be sufficient to explain the experiments.

The **Author**, in reply to Mr. Appleyard, said the first two sections of the Paper referred to two entirely different sets of measurements, and the main object of the second part was to show that the presence of a discontinuity between the dielectric and its electrodes had no very appreciable effect upon the measurement of the power absorbed by the condenser due to the *alternating current* conductivity, which was something quite different from the conductivity for direct currents. The point had been raised as to whether the capacity changes in Table VII. were due to the mechanical pressure. It was found that the resistance of the celluloid, measured with the full pressure on, was practically the same as when measured with mercury electrodes, and that showed that there was no sensible decrease of thickness with pressure, and that the increase of capacity was not, therefore, due to the effect of mechanical pressure. In reply to Dr. Russell, he said the representation given in the Paper of the condenser with its imperfectly travelling armatures only concerned the alternate current losses. The purely leakage losses could easily be separated out in the manner described. Mr. Addenbrooke had referred to the large temperature coefficient of the conductivity of celluloid, but this had no application to the case of part three, because the voltage applied to the bridge was only of the order of 2 or 3 volts. Mr. Duddell's remarks were very interesting, and showed the considerable influence of brush discharge even at low voltages. In reply to Prof. Lees, tinfoil electrodes under pressure had not been tried in the residual charge experiments, but it was to be expected that under those circumstances we should get a curve intermediate between curves 4 and 5 of Fig. 10. No doubt, as Dr. Eccles suggested, a combination of the two effects described in the first and last parts of the Paper would account for the viscous part of the phenomenon.

XXXV. *On a Method of Measuring the Pressure of Light by Means of Thin Metal Foil.* By GILBERT D. WEST, *B.Sc.*

RECEIVED MAY 29, 1913.

IF a flexible uniform strip of foil suspended vertically be subjected to a small horizontal pressure, the consequent deflection θ is given by

$$\theta = R/\rho,$$

where R is the horizontal force per unit area and ρ is the weight of unit area of the foil. The pressure of bright sunlight is about 5×10^{-8} gramme, and the mass of a square centimetre of gold leaf is about $1{\cdot}6\times10^{-4}$ gramme. The end of a strip 10 cm. long should, therefore, be deflected by 0·003 cm., a distance easily observable with a microscope. In the following the above method is developed, and experiments with both gold and aluminium foil are described. The interest of the research lies in the extreme simplicity of the apparatus and in the absence of complicated adjustments. To some extent the experiments resemble those carried out by Prof. Osborne-Reynolds on the impulsion of silk fibres,* but the latter's experiments had for their object an investigation of the gas action.

The Apparatus.

Various methods of mounting the strips of foil were tried, but the method of making a loop at the top was found to be the most satisfactory.

A piece of gold or aluminium foil was bent over upon itself, covered with tissue paper, and the doubled portion then struck at two or three points with a very small round-headed hammer. This was sufficient to make the two pieces of foil adhere and a strip with a loop at the top was thus formed.

A glass tube was then bent into the shape indicated by Fig. 1. AB is a capillary tube, and C is a mercury manometer. It was found an advantage to dust the portion AB with a fine powder, such as red ochre, for otherwise the foil tended to adhere to the glass. The strip was then mounted in a tube (previously dusted with red ochre) as shown in Fig. 1. Three such strips were prepared for the final measurements, two of aluminium and one of gold. The method of mounting was found to be very satisfactory, and whether the strip was deflected by

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* Osborne-Reynolds, "Phil. Trans.," Part II., 1879, p. 768.

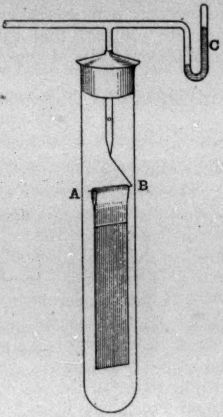


FIG. 1.

rotation of the tube about an axis AB, or by outside electrical attraction the return to the eyepiece zero of the observing microscope was always good.

Source of Radiation.

As a source of radiation a 110-watt carbon filament lamp was used. This lamp, without concentrating lenses but sometimes with a glass screen in front of it, was placed at distances from the strips varying from 12 cm. to 16 cm. It was found necessary to enclose the lamp in an earthed tin box with a hole in the front, as without this deflections could sometimes be obtained by simply charging the filament to 200 volts.

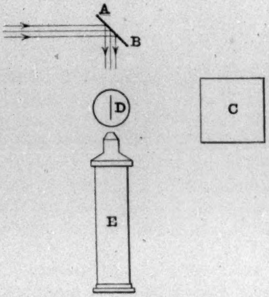
Observing Microscope.

In the observing microscope, use was made of a × 20 eyepiece coupled with an inch objective. By this means, defined magnification of about 200 could be obtained, together with the long working distance essential to the observations.

Weight of the Strips.

The weights of the strips were determined by the bending they produced in a fibre of glass that had been previously calibrated with small pieces of wire of known weight.

Arrangement of Apparatus.



AB is a piece of glass coated with foil reflecting light to illuminate field, C is the source D the strip of foil in tube and E the observing microscope.

FIG. 2.

Miscellaneous Facts.

Calculation shows that the heating of the strip reaches the steady state in a fraction of a second. Inasmuch as the side nearer the source is heated most, there is a small deflection due to the consequent curvature. It can be shown, however, that such a deflection is only about $\frac{1}{1000}$ the deflection produced by the pressure of radiation.

Light incident on a gold or aluminium strip is reflected without much scattering. From experiments carried out with a thermometer, whose bulb was coated with foil, it was found that both gold and aluminium leaf reflected about 84 per cent. of the incident light. The pressure on a square centimetre of foil should thus be equal to 1·84 times the energy per cubic centimetre.

Calculation of the Pressure.

By taking moments about an axis AB we find that R, the radiation pressure, is given by

$$R = \frac{2 \, . \, Whd}{Al^2},$$

where W is the weight of the strip (in dynes),

h is the distance of the centre of gravity of the strip below AB,

d is the deflection as measured by the observing microscope,

A is the exposed area of the strip,

and l is its length.

RESULTS OBTAINED.

Air at Atmospheric Pressure.

The general effect of the incidence of radiation upon the front of a strip is a small motion backwards (sometimes of the order of the deflection required by the pressure of radiation) followed by a more powerful motion forward due to convection currents. Sometimes the motion is a little unsteady at first, but the forward motion due to convection always takes place finally. The magnitude of the convection effect is much reduced by placing a glass screen in front of the lamp, and this reduction is doubtless caused by the decreased heating of the walls of the test tube. No satisfactory observations can be taken with air at atmospheric pressure.

Air at Lower Pressures.

As the pressure is reduced so does the forward convection effect get gradually less, until, at a pressure of a few centimetres of mercury it is not observable. At pressures of about 1 to 2 cm. of mercury, very small convection effects still occur, but the fortunate fact about these effects is that they always act on the same side of the strip and are independent of the side on which the source is placed. Similar results to these were obtained by Hull* in 1905. The effects are probably attributable to the small inclination of the strip to the vertical and to the consequent results of the strip being heated. An example is given below.

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* Hull, "Phys. Review," XX., 1905, p. 292.

Gold strip. Air 1·3—1·6 cm. Hg.		
Distance of source.	Deflections in eyepiece divisions.	
	Source on left.	Source on right.
12 cm.	0·8 towards source	2·9 from source
14 cm.	0·7 towards source	2·1 from source
16 cm.	0·6 towards source	1·6 from source

When the source was on the left the two effects were in opposition, but when the source was on the right they were in conjunction. The true radiation pressure could thus be calculated.

Hydrogen at Atmospheric Pressure.

It is well known that convection effects in hydrogen are much less than in air.* When the tube was filled with hydrogen at atmospheric pressure the convection effects were of the same kind but somewhat smaller than with air at 1 to 2 cm. pressure. An example is given below.

Gold strip. Hydrogen at atmospheric pressure.		
Distance of source.	Deflections in eyepiece divisions.	
	Source on left.	Source on right.
12 cm.	0·7 from source	1·4 from source
14 cm.	0·5 from source	1·2 from source
16 cm.	0·3 from source	1·0 from source

Hydrogen at Low Pressures.

With hydrogen at low pressures the effects obtained were of the same kind as with air at low pressures, but they were somewhat smaller. An example is given below.

Gold strip. Hydrogen 1·3—1·5 cm. Hg.		
Distance of source.	Deflections in eyepiece divisions.	
	Source on left.	Source on right.
12 cm.	0·0 from source	2·0 from source
14 cm.	0·0 from source	1·5 from source
16 cm.	0·0 from source	1·1 from source

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* Barlow, "Proc." Royal Soc., A., Vol. LXXXVII., 1912, p. 1.

Measurement of the Energy per Cubic Centimetre.

The measurement of the energy received per square centimetre per second at various distances from the source was made by observing the initial rates of rise of temperature of a blackened copper plate 3 mm. thick enclosed in a test tube. The temperature of the copper plate was indicated by an embedded copper-eureka thermojunction connected to a low resistance D'Arsonval galvanometer. The lampblack was assumed to absorb 95 per cent. of the incident radiation.

—	Distance of source.	Energy per Cubic Centimetre.	Pressure of radiation as calculated from the deflections of					
			Gold strip. Width=1·43 cm. Length=6·4 cm. Containing tube=2·4 cm. diameter.			Aluminium strip. Width=1·11 cm. Length=7·64 cm. Containing tube=1·6 cm. diameter.		
			Hydrogen pressure atmospheric.	Hydrogen pressure 1–2 cm. Hg.	Air pressure 1–2 cm. Hg.	Hydrogen pressure atmospheric.	Hydrogen pressure 1–2 cm. Hg.	Air pressure 1–2 cm. Hg.
		Ergs.	Dynes.	Dynes.	Dynes.	Dynes.	Dynes.	Dynes.
Glass off	12 cm.	$3{\cdot}2 \times 10^{-5}$	$3{\cdot}3 \times 10^{-5}$	$3{\cdot}5 \times 10^{-5}$	$4{\cdot}0 \times 10^{-5}$	$3{\cdot}7 \times 10^{-5}$	$2{\cdot}5 \times 10^{-5}$	$3{\cdot}5 \times 10^{-5}$
	14 cm.	2·5	2·7	2·6	2·9	3·0	2·3	3·2
	16 cm.	1·9	1·8	2·0	2·4	2·6	2·0	2·3
Glass on	12 cm.	2·5	2·9	2·5	2·7	2·6	2·9	2·9
	14 cm.	1·8	2·2	1·9	1·8	2·1	2·1	1·8
	16 cm.	1·4	1·7	1·4	1·3	1·5	1·5	1·3

An exactly similar set of results was obtained with an aluminium strip 1·66 cm. wide, 7·3 cm. long and enclosed in a tube 2·4 cm. diameter. The fact that the majority of results are rather high suggests the existence of either a radiometer effect or some constant error in one of the measuring instruments.

The measurement of the radiation pressure by the means described is so simple that there is now no reason why it should not be included in a course of laboratory experiments.

In conclusion, I desire to thank Prof. Lees for his encouragement and suggestions and for the facilities placed at my disposal at East London College.

ABSTRACT.

The pressure of the radiation emitted by a carbon filament lamp at a distance of a few centimetres is sufficient to cause a microscopically measurable deflection of the end of a suspended strip of gold or aluminium foil, and by this means the radiation pressure can

be calculated knowing the weight of the strip. The results agree to within about 10 per cent. with the energy content per cubic centimetre as measured by the initial rate of rise of temperature of a copper plate exposed to the radiation.

The best results are obtained by working in an atmosphere of hydrogen, 1 cm. to 2 cm. pressure, but good results are obtained with hydrogen at atmospheric pressure. Air at 1 cm. to 2 cm. pressure also gives good results.

The method involves no laborious adjustments, and the apparatus is not seriously affected by vibration.

DISCUSSION.

Prof. C. H. **Lees** remarked that the point in the above Paper was its extreme simplicity.

Dr. G. **Barlow** (in some remarks communicated by Prof. Poynting) pointed out that the author ought to allow for the light reflected from the foil to the glass wall and back again to the foil. This would improve the agreement in the author's measurements.

XXXVI. *The Quantum Theory of Energy and the Emission of Electricity from Hot Bodies.*

By WILLIAM WILSON, *Ph.D.*,
Wheatstone Laboratory, King's College, London.

RECEIVED JUNE 9, 1913.

THE quantum theory was introduced by Planck* in order to arrive at a formula representing the distribution of energy in the spectrum of the radiation from a black body. For the purposes of the present Paper Planck's formula may be put in the form

$$I_\nu d\nu = C\nu^3 \frac{d\nu}{e^{\frac{h\nu}{kT}} - 1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $I_\nu d\nu$ represents the quantity of energy within the frequency limits ν and $\nu + d\nu$ emitted per second by the "resonators" of the radiating material, T is the absolute temperature, C is a suitable constant and h and k are universal constants of nature. Comparison of the results of measurement on the radiation of a black body with this formula and with the Boltzmann-Stefan formula for the total radiation from a black body give for these constants the values

$$h = 6{\cdot}415 \,.\, 10^{-27} \text{ erg-seconds,}$$

$$k = 1{\cdot}34 \,.\, 10^{-16} \text{ ergs per degree.}$$

The latter constant is identical with the absolute gas constant reckoned for one molecule. It is possible, therefore, to deduce from measurements on black radiation the number of molecules of an ideal gas per cubic centimetre under standard conditions of pressure and temperature. Planck finds this number to be

$$2{\cdot}77 \,.\, 10^{19}.$$

A further consequence is the deduction of the elementary charge of electricity. This can be deduced from this last number, and the results of electrolysis, and is found to be

$$e = 4{\cdot}67 \,.\, 10^{-10} \text{ electrostatic units.}$$

In deducing his formula, Planck assumes that the radiating body contains a number of "resonators" which emit their energy in a discontinuous way. Let us suppose, for example, that one of these "resonators" at a certain instant has no energy. It will, on Planck's hypothesis, absorb energy continuously from the surrounding radiation. When its energy reaches the value $h\nu$, where ν is the frequency of the "resonator" and h the constant already mentioned, it *may* emit the whole of its energy. If emission does not occur, it will continue to absorb energy till the amount $2h\nu$ has been acquired, when the emission of the whole of its energy may occur, and so on. Emission *only* occurs when the energy of the "resonator" is an integral multiple of $h\nu$, the actual occurrence of the emission being a matter of chance.

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* "Ann. d. Physik," Bd. 4, p. 553 (1901); Bd. 4, p. 564 (1901); "Theorie der Wärmestrahlung." Second edition.

Not only does Planck's formula represent accurately the distribution of energy in the black spectrum, but the hypothesis of discontinuous emission which was used by Planck to deduce the formula seems to admit of a very wide application. Hughes,* for example, concludes that the energy of electrons emitted from metals illuminated with ultra-violet light is equal to a constant multiplied by the frequency of the light, and this constant is very nearly equal to Planck's h. Prof. O. W. Richardson and Mr. Compton† have also arrived at similar conclusions. Further, the value recently obtained by Prof. Barkla and Mr. G. H. Martyn‡ for the wave-length of the Röntgen radiation used in their experiments closely accords with the value obtained from the equation

$$\tfrac{1}{2}mv^2 = h\nu, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

when we substitute for m and v the mass and velocity respectively of the electrons which would be emitted if the radiation were absorbed by a metal, and for h the value given by Planck.

The above considerations justify an attempt to apply Planck's theory to the phenomena of emission of electrons from hot bodies, and they suggest that emissions of energy take the form of ejection of electrons. Prof. Nicholson§ has also adopted this hypothesis in his work on the constitution of the solar corona.

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* Hughes, "Phil. Trans.," Vol. CCXII. (A), p. 205 (1912), and "Phil. Mag.," Vol. XXV., p. 683 (1913).

† Richardson and Compton, "Phil. Mag.," p. 576 (1912).

‡ Barkla and Martyn, "Proc. Phys. Soc. of London," Vol. XXV., p. 214 (1913).

§ "The Constitution of the Solar Corona II.," Monthly Notices of R.A.S., June, 1912.

Let us suppose we have a hot body—*e.g.*, platinum—in a field of black radiation. The emission of the energy

$$I_\nu d\nu = C\nu^3 \frac{d\nu}{e^{\frac{h\nu}{kT}}-1}$$

is accomplished by the ejection of dN electrons. Each electron at the moment of emission from the resonator or atom has a quantity of energy equal to an integral multiple of $h\nu$, so that we may write

$$C\nu^3 \frac{d\nu}{e^{\frac{h\nu}{kT}}-1} = \bar{n}h\nu dN, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $\bar{n}$ is the average value of the integer in question. If we put $\varepsilon = h\nu$, and suitably modify the constant C, the last equation will take the form

$$C\varepsilon^2 \frac{d\varepsilon}{e^{\frac{\varepsilon}{kT}}-1} = \bar{n}dN. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\text{A})$$

Most of these dN electrons probably never leave the molecule within which they are emitted at all, but simply leave one atom and attach themselves to another. In doing so they contribute $I_\nu d\nu$ to the radiant energy within the frequency limits ν and $\nu + d\nu$.

The following considerations will assist us in dealing with the number $\bar{n}$. If the probability of an emission within the frequency range ν to $\nu + d\nu$ is very great, $\bar{n}$ will approach unity. On the other hand, if this probability is very small $\bar{n}$ may be very great. The simplest hypothesis we can introduce to connect $\bar{n}$ and the probability η of an emission is that expressed by the equation

$$\bar{n} = \frac{1}{\eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Planck finds for η the value

$$\eta = \frac{e^{\frac{\varepsilon}{kT}}-1}{e^{\frac{\varepsilon}{kT}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (3*a*), (4) and (5) we get the very simple equation

$$C\varepsilon^2 e^{-\frac{\varepsilon}{kT}} d\varepsilon = dN. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The number of electrons emitted by the "resonators" whose energy exceeds a certain value w will be given by

$$N = C\int_w^\infty \varepsilon^2 e^{-\frac{\varepsilon}{kT}} d\varepsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It is likely that only electrons from the surface layer of molecules leave the metal, and, if w is the smallest amount of energy an electron can have in order to leave a molecule in the surface layer, equation (7) will, if a suitable value is given to C, represent the quantity of electricity emitted per second from a square centimetre of the surface of the metal. On integrating (7) we get the formula

$$Q=KT\left(1+2\frac{k}{w}T+2\frac{k^2}{w^2}T^2\right)e^{-\frac{w}{kT}}, \quad . \quad . \quad . \quad . \quad (8)$$

where K is a suitably chosen constant.

The following table gives the leak observed by H. A. Wilson* from a platinum wire which had been treated with nitric acid, and also the leak as calculated from O. W. Richardson's formula

$$Q=aT^{\frac{1}{2}}e^{-\frac{w}{kT}},$$

$$a=6\cdot9\,.\,10^7,$$

$$\frac{w}{k}=6\cdot55\,.\,10^4.$$

In the third column is given the leak as calculated from formula (8). The constants w and K were calculated by making the formula agree with the extreme values 15·7 at 1,375°C. and 1,280 at 1,580°C. For w/k the value 6·376 . 10^4 was used. The term $2\frac{k^2}{w^2}T^2$ was neglected in the calculation, since it would only affect the result by less than $\frac{1}{4}$ per cent. if the temperature were 2,000° abs., and, of course, to a smaller extent at lower temperatures.

Temperature in degrees Centigrade.	Leak observed per square centimetre.	Leak calculated by formula (8).	Leak calculated by Richardson's formula.
1,375	15·7	15·7	14·9
1,408·5	34·3	34·6	33·3
1,442	74·6	74·1	71·8
1,476	152	155·7	153
1,510·5	323	321·8	318
1,545	638	647	645
1,580	1,280	1,280	1,285

---

* H. A. Wilson, "Phil. Trans.," Vol. CCII. (A), p. 258 (1904). J. J. Thomson, "Conduction of Electricity through Gases," p. 202, second edition. The unit of current used in the table is 10^{-9} amperes.

It will be seen that formula (8) is in distinctly better accordance with the experimental values than is Richardson's formula.

The foregoing may be briefly summarised in the following way. A definite theory has been proposed for the phenomena of emission of electricity from hot bodies. This theory is based on certain hypotheses as to the manner in which interchange of energy between a radiating body and the surrounding radiation occurs. The assumption that energy is emitted by the ejection of electrons each carrying an integral multiple of $h\nu$ units of energy has led to a formula which represents the facts better than that hitherto used, while it has the important advantage of being applicable to a wide range of phenomena. It has been used with great success in accounting for the origin of the coronal spectral lines and is in accordance with the latest experimental results on photo-electric phenomena and Röntgen rays.

3RD JUNE, 1913.

ABSTRACT.

The Paper gives a theory of the emission of electricity from hot bodies which is based on the quantum theory of energy. A formula connecting the thermionic current and the temperature of the emitting body is deduced. This formula closely resembles that of Richardson, and agrees slightly better with experimental results.

DISCUSSION.

Prof. J. W. **NICHOLSON** thought the Paper was a valuable one in that it connected up yet another phenomenon with Planck's quantum theory of radiation. It was probable that Planck's constant h was in some way an electron constant, and that the emission of energy was discontinuous because emission of electrons was discontinuous.

XXXVII. *Note on the Resistivities of Glass and Fused Silica at High Temperatures.* *By* ALBERT CAMPBELL, *B.A.*

(From the National Physical Laboratory.)

RECEIVED JUNE 18, 1913.

SOME years ago (1906) we made some tests of the insulation resistance of fused quartz, glass and mica at various temperatures. As the resistivities of such materials have not yet been thoroughly investigated, I give below (by the kind permission of the Thermal Syndicate) some of the results obtained in our experiments. They must be taken as giving the order of magnitude of the resistivities rather than highly accurate numbers, for the walls of some of the tubes were somewhat irregular, and no great elaboration of apparatus or methods was employed. In the case of the silica and glass the tests were made on tubes about 30 cm. long, 1 cm. to 4 cm. diameter, and of wall thicknesses between 0·05 mm. and 0·12 mm.

The mean thicknesses were obtained from the density and the mass, due estimates being made of the effect of want of uniformity.

The tubes were platinized inside and outside for a considerable part of their length. Towards each end beyond the platinized surfaces there were bare strips, and beyond these were platinized bands, to which guard wires were connected. All the connections to the platinized surfaces were made by means of nickel wires.

Most of the glass tubes were entirely closed at one end, so as to require only one guard strip. The guard wires were arranged in the well-known manner to avoid all surface leakage. For the high temperatures the tubes were heated in a suitable electric furnace, and the temperature was measured by a calibrated thermocouple.

The resistance measurements were made by the direct deflection method, using a galvanometer with a set of shunts. The readings were taken after a time of electrification of one minute, and the voltage employed was either 200 or 500 volts, except for a few of the readings at the highest temperatures, which were taken with 2 volts.

The mica was tested in the form of a thin sheet, with platinized areas on each face and a rim for the guard wire.

In the following table are given the results, which have in some cases been averaged from several experiments. It will be noticed that at 750 deg. the Jena combustion glass conducted fairly well. With 2 volts the apparent resistance of the piece tested rose quickly from 150 to 600 ohms owing to polarization.

Material.	Description.	Temperature, deg, C.	Resistivity, megohm-cm.
Fused silica ...	Silky surface.........	15	Over 200,000,000
		150	Over 200,000,000
		230	20,000,000
		250	2,500,000
		300	200,000
		350	30,000
		450	800
		700	30
		800	About 20
		850	About 20
Glass	Ordinary (soda-lime)	18	500,000
		145	100
Glass	" Geräte " (zinc-aluminium)......	18	3,000,000
Glass	Jena (combustion tubing)	18	Over 200,000,000
		115	36,000,000
		150	18,000,000
		750	0·01 to 0·04
Mica	0·026 mm. thick ...	18	Over 300,000,000
		135	Over 300,000,000

It is clear from the above somewhat meagre figures that the subject would repay much fuller investigation. In the case of mica the resistivity is reduced very considerably at the higher temperatures, but still not nearly as much as for the other materials mentioned above.

June 5, 1913.

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